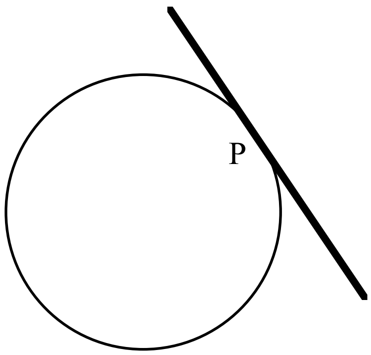
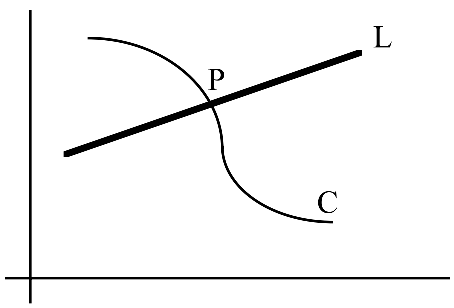
Tangent Lines - Classwork



In plane geometry, we say that a line is tangent to a circle if it intersects the circle at one point. However, for more general curves, we need a better definition. The idea of tangent lines is crucial to your understanding of differential calculus so we must have an accurate idea of its meaning. There are many "rough" ideas of what a tangent line is; they are not only rough but wrong. Without looking at the rest of the page, write below what your definition of what a tangent line to a curve is.

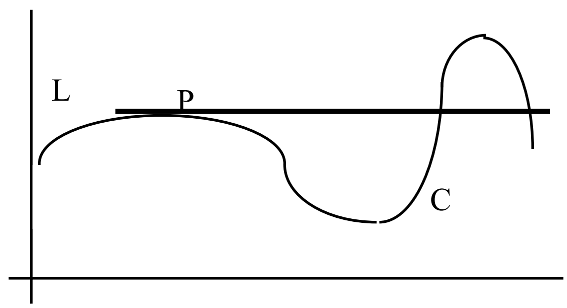
____

____



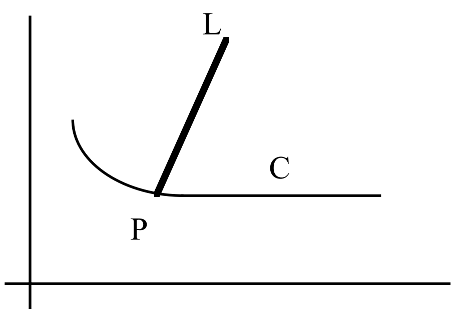
Misconception 1: "A line is tangent to a curve if it crosses the curve at exactly one point."

This is wrong. Line L touches curve C at only one point P and is *not* a tangent line.



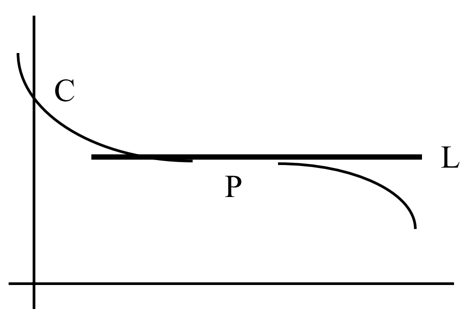
Misconception 2: "A tangent line to a curve must cross the curve only once."

This is also wrong. Line L is tangent to curve C at point P but it also crosses curve C at 2 other points



Misconception 3: "A line is tangent to a curve if touches the curve at one point but does not cross the curve."

Again, here is a counterexample to that statement. Line segment L touches curve C at point P but is clearly not tangent to C at P.



Misconception 4: "A tangent line to a curve is a line that just "grazes" the curve at a point but does not cross the curve."

Still another counterexample. Line L *is* tangent to curve C at point P but does cross the curve."

It will be some time before we can get a clear definition of a tangent line. At this point, we simply wish to give you some inkling of the difficulty of creating such a definition. We will have to rely on our general knowledge to draw a tangent line.

For each equation, graph in an appropriate window and draw the tangent line at the indicated point.

1) $y = -2x \quad (x = -1)$

Scale [,] [,]

2) $y = 1 - x^2 \quad (x = 0)$

Scale [,] [,]

3) $y = x^3 - 3x^2 + 3x - 1 \quad (x = 1)$

Scale [,] [,]

4) $y = x^2 - x^4 \quad (x = 1)$

Scale [,] [,]

5) $y = |3 - x| \quad (x = 3)$

Scale [,] [,]

6) $y = e^x \quad (x = 0)$

Scale [,] [,]

7) $y = \sin x \quad (x = \pi)$

Scale [,] [,]

8) $y = \sqrt{4 - x} \quad (x = 4)$

Scale [,] [,]

Tangent Lines - Homework

For each equation, graph in an appropriate window and draw the tangent line at the indicated point.

1. $y = x^2 \quad (x = 2)$

Scale [,] [,]

2. $y = \dfrac{x}{3} \quad (x = -2)$

Scale [,] [,]

3. $y = x^3 \quad (x = 0)$

Scale [,] [,]

4. $y = x^4 - x^2 \quad (x = -1)$

Scale [,] [,]

5. $y = |x| \quad (x = 0)$

Scale [,] [,]

6. $y = 2^x \quad (x = 0)$

Scale [,] [,]

7. $y = \cos x \quad \left(x = \dfrac{\pi}{2}\right)$

Scale [,] [,]

8. $y = \sqrt{x} \quad (x = 0)$

Scale [,] [,]

9. $y = \sqrt[3]{x} \quad (x = 0)$

Scale [,] [,]

10. $y = x^{2/3} \quad (x = 0))$

Scale [,] [,]

11. $y = \dfrac{x}{x-2} \quad (x = 2)$

Scale [,] [,]

12. $y = \tan x \quad \left(x = \dfrac{\pi}{2}\right)$

Scale [,] [,]

13. $y = \ln x \quad (x = e)$

Scale [,] [,]

14. $y = x + \dfrac{1}{x} \quad (x = 1)$

Scale [,] [,]

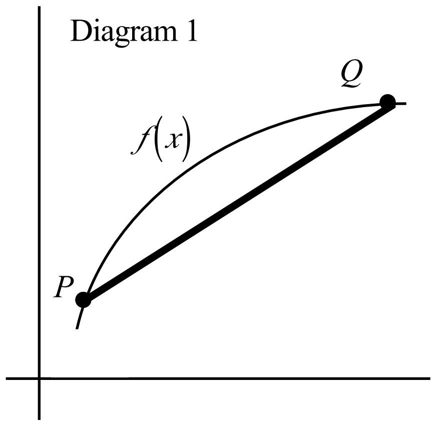
15. $y = \sqrt{100 - x^2} \quad (x = 0)$

Scale [,] [,]

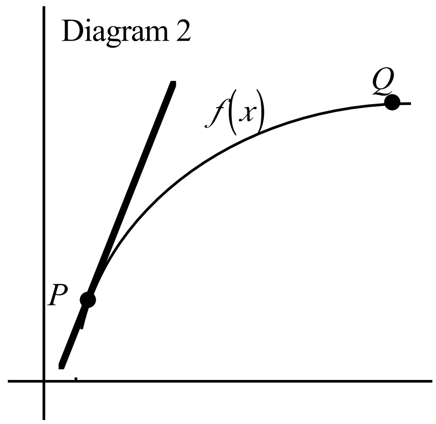
16. $y = \sqrt{100 - x^2} \quad (x = -10)$

Scale [,] [,]

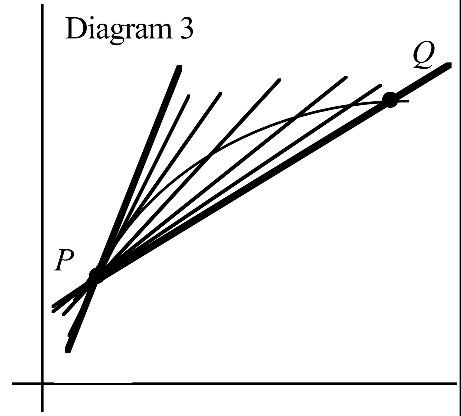
Slopes of Secant and Tangent Lines - Classwork



A line is drawn between points P and Q. That line is called the **secant line through P and Q.**



A line is drawn through P that touches $f(x)$ in **one and only one point**. That line is called the **tangent line at P.**



We draw the secant line through PQ. However, point Q starts to move along the curve $f(x)$ towards P. So the secant line also moves. As Q gets real close to P note that the secant line starts to look like the tangent line at P.

We will say that as Q gets closer and close to P, that the secant line PQ gets closer to the tangent line through P and thus the slope of the secant line (called m_{sec}) approaches the slope of the tangent line (called m_{tan}).

Let's try an example. Suppose $f(x) = x^2$. Let us try and find the slope of the secant line between $x = 1$ and a value of x closer and closer to 1. Complete the chart. Set your calculator to full float.

x	**2**	**1.5**	**1.1**	**1.05**	**1.01**	**1.001**	**1.0001**	**1**
$f(x)$								
Rise								
Run								
m_{sec}								

Why can't you find the tangent line between $x = 1$ and $x = 1$? ______________________________

Secant lines use how many points? ____________ Tangent lines use how many points? ____________

What is your best guess for the slope of the tangent line at $x = 1$? ____________________

Let us use an analogy. Suppose you leave school at 10 AM for a trip down the shore. You arrive at 12 PM. The trip is a total of 100 miles. How fast are you going at 11 AM? ______________________

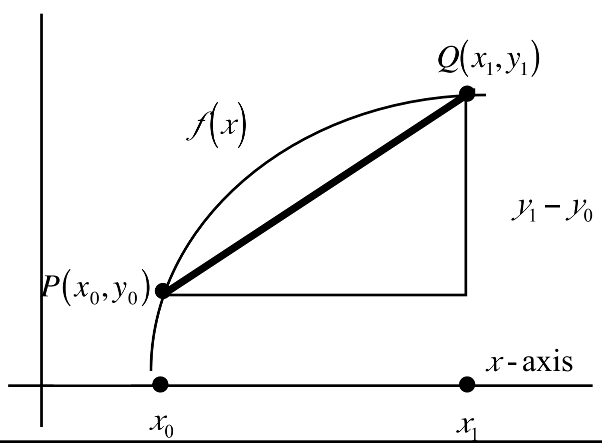
We cannot find your actual velocity at 11 AM (called the **instantaneous velocity**). But we can find the **average velocity** at 11 AM. We know that $\text{average velocity} = \dfrac{\text{total distance}}{\text{total time}}$. So your average velocity between 10 AM and 12 PM is _______mph. On the next page, you are given the distance you travel between 11 AM and different times. Determine the average velocity between these two times.

between 11 am &	11:30	11:15	11:10	11:05	11:02	11:01	11:00:30	11:00:01	11:00
distance traveled	24 miles	13 miles	10 miles	5.5 miles	2 miles	.8 miles	.53 miles	80 ft.	_____
time duration									
average velocity									
velocity at 11 AM									

Again, you cannot actually find the instantaenous velocity at 11 AM? Why? ______________________________

But, as the time duration becomes smaller we find that the instantaneous velocity at 11 AM is much more likely to be very close to which of these values? ____________ Why? ______________________________________

Average velocity uses how many times? _________ Instantaneous velocity uses how many times? ___________



Let us now take the two points and give them general coordinates $P(x_0, y_0)$ and $Q(x_1, y_1)$

We draw the right triangle below the curve. Note that the length of the horizontal line is $x_1 - x_0$ and the length of the vertical line is $y_1 - y_0$

We now find the slope of the secant line PQ and denote it as m_{sec}.

$$m_{\text{sec}} = \frac{\text{rise}}{\text{run}} = \frac{y_1 - y_0}{x_1 - x_0}$$

However, since another way of writing *y* is $f(x)$, we can say that $\text{m}_{\text{sec}} = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$

Now, let's concentrate on the tangent line. In order to find the slope of the tangent line, the above formula does not work. Why? ____.

So, let us define the variable *h* as the horizontal distance between the two points P and Q. Thus:

$h = x_1 - x_0$ and it follows that $x_1 = x_0 + h$

Since $\text{m}_{\text{sec}} = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$, it follows that $\text{m}_{\text{sec}} = \dfrac{f(x_0 + h) - f(x_0)}{h}$

THE KEY: Remember that the tangent line on the previous page was defined as the line created when Q gets closer and closer to P. As Q gets closer and closer to P, x_1 gets closer and closer to x_0. So, as Q gets closer to P *h* (the horizontal distance between P and Q) gets close to zero.

Therefore, we can now state the starting point for differential calculus:

$\text{m}_{\tan} = \dfrac{f(x_0 + h) - f(x_0)}{h}$ as *h* gets infinitely close to zero. Note that *h* cannot equal zero. Why not?

Three formulas you will need to know are:

- the slope of the secant line: $m_{sec} = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$
- the slope of the tangent line: $m_{tan} = \dfrac{f(x_0 + h) - f(x_0)}{h}$ as h gets infinitely close to zero.
- the point-slope equation of a line: $y - y_1 = m(x - x_1)$

Example 1) For the function $f(x) = x^2 + 1$, find the following. Confirm c) on your calculator.

a) the slope of the secant line between $x = 1$ and $x = 3$

b) the slope of the tangent line at $x = 2$.

c) the equation of the tangent line at $x = 2$.

Example 2) For the function $f(x) = 5x + 2$, find the following. Confirm c) on your calculator.

a) the slope of the secant line between $x = -1$ and $x = 4$

b) the slope of the tangent line at $x = 2$.

c) the equation of the tangent line at $x = 2$.

Example 3) For the function $f(x) = x^2 + 4x - 1$, find the following. Confirm c) on your calculator.

a) the slope of the secant line between $x = -3$ and $x = -1$

b) the slope of the tangent line at $x = -2$.

c) the equation of the tangent line at $x = -2$.

Example 4) For the function $f(x) = 2x^2 - 5x - 3$, find the following. Confirm c) on your calculator.

a) the slope of the secant line between $x = 0$ and $x = 1$

b) the slope of the tangent line at $x = 2$.

c) the equation of the tangent line at $x = 2$.

Example 5) For the function $f(x) = x^3 - x^2 + 1$, find the following. Confirm c) on your calculator.

a) the slope of the secant line between $x = -1$ and $x = 1$

b) the slope of the tangent line at $x = 1$.

c) the equation of the tangent line at $x = 1$.

Example 6) For the function $f(x) = \dfrac{2}{x+1}$, find the following. Confirm c) on your calculator.

a) the slope of the secant line between $x = 1$ and $x = 4$

b) the slope of the tangent line at $x = 2$.

c) the equation of the tangent line at $x = 2$.

Let us suppose that an object is traveling along a straight line according to the formula $s(t) = 2t + 3$ where t is measured in seconds and $s(t)$ is measured in feet. Complete the chart.

t	0	1	2	3	4	5
$s(t) = 2t + 3$						

If we want to calculate the average velocity between $t = 0$ and $t = 4$, we know $\text{average velocity} = \dfrac{\text{total distance}}{\text{total time}}$

So, average velocity equals ________________ measured in ________________.

But, if we wish to calculate the instantaneous velocity at $t = 3$ seconds, we are interested in exactly how fast we are traveling at $t = 3$. This is not as easy to do.

Using the analogy above, we can now state 2 formulas which allow us to find both average and instantaneous velocity.

> Two formulas you will need to know are: Given $s(t)$ as the distance traveled in time t
>
> - Average velocity = $\dfrac{s(t_2) - s(t_1)}{t_2 - t_1}$
> - Instantaneous velocity: $\dfrac{s(t_1 + h) - s(t_1)}{h}$ as h gets infinitely close to zero.

So in the problem above, let's find the instantaneous velocity at $t = 3$.

Note that the average velocity is the same as the instantaneous velocity. If you are in a car where your average speed is the same as your instantaneous speed, what is that called? ________________________________

Example 6) If $s(t) = 3t + 1$ is a measure of feet traveled per second, find,

a) the average velocity between $t = 0$ and $t = 3$

b) the instantaneous velocity at $t = 2$ seconds.

Example 7) If $s(t) = t^2 - 2t$ is a measure of feet traveled per second, find

a) the average velocity between $t = 0$ and $t = 2$

b) the instantaneous velocity at $t = 2$ seconds.

Example 8) If $s(t) = t^3 + t^2 - t - 1$ is a measure of feet traveled per second, find

a) the average velocity between $t = 1$ and $t = 2$

b) the instantaneous velocity at $t = 1$ second.

Slopes of Secant and Tangent Lines - Homework

1. For the function $f(x) = 5x + 3$, find the following. Confirm c) on your calculator.

   a) the slope of the secant line between $x = 1$ and $x = 5$

   b) the slope of the tangent line at $x = 2$.

   c) the equation of the tangent line at $x = 2$.

2. For the function $f(x) = x^2 - 3$, find the following. Confirm c) on your calculator.

   a) the slope of the secant line between $x = 0$ and $x = 3$

   b) the slope of the tangent line at $x = 1$.

   c) the equation of the tangent line at $x = 1$.

3. For the function $f(x) = x^2 - 5x + 4$, find the following. Confirm c) on your calculator.

   a) the slope of the secant line between $x = 2$ and $x = 6$

   b) the slope of the tangent line at $x = 3$.

   c) the equation of the tangent line at $x = 3$.

4. For the function $f(x) = 2x^2 - 7x + 8$, find the following. Confirm c) on your calculator.

   a) the slope of the secant line between $x = -1$ and $x = 4$

   b) the slope of the tangent line at $x = -1$.

   c) the equation of the tangent line at $x = -1$.

5. For the function $f(x) = x^3 + x - 2$, find the following. Confirm c) on your calculator.

a) the slope of the secant line between x = -3 and x = 3

b) the slope of the tangent line at x = 2.

c) the equation of the tangent line at x = 2.

6. For the function $f(x) = \frac{5}{x-3}$, find the following. Confirm c) on your calculator.

a) the slope of the secant line between x = 4 and x = 6

b) the slope of the tangent line at x = 1.

c) the equation of the tangent line at x = 1.

7. For the function $f(x) = \frac{x-2}{x+1}$, find the following. Confirm c) on your calculator.

a) the slope of the secant line between x = 1 and x = 5

b) the slope of the tangent line at x = 3.

c) the equation of the tangent line at x = 3.

8. If $s(t) = 4t + 1$ is a measure of feet traveled per second, find
a) the average velocity between $t = 1$ and $t = 5$
b) the instantaneous velocity at $t = 2$ seconds.

9. If $s(t) = t^2 + 4$ is a measure of feet traveled per second, find
a) the average velocity between $t = 0$ and $t = 4$
b) the instantaneous velocity at $t = 1$ second.

10. If $s(t) = t^2 - 3t + 2$ is a measure of miles traveled per hour, find
a) the average velocity between $t = 0$ and $t = 4$
b) the instantaneous velocity at $t = 1$ hour.

11. If $s(t) = t^3 + t - 1$ is a measure of feet traveled per second, find
a) the average velocity between $t = 2$ and $t = 7$
b) the instantaneous velocity at $t = 2$ seconds.

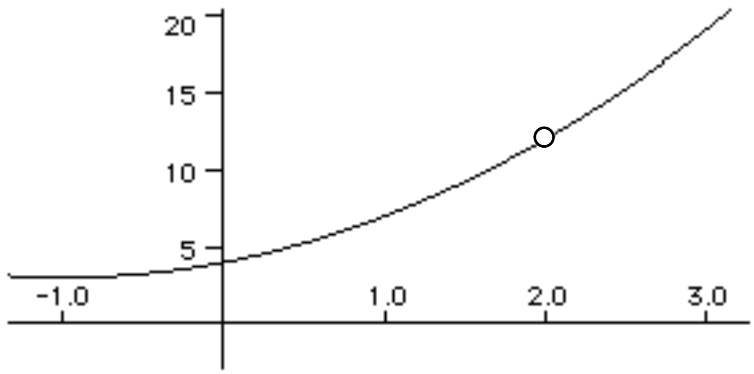
12. If $s(t) = \dfrac{6}{t + 2}$ is a measure of feet traveled per second, find
a) the average velocity between $t = 1$ and $t = 7$
b) the instantaneous velocity at $t = 4$ seconds.

Graphical Approach to Limits - Classwork

Suppose you were to graph

$$f(x)=\frac{x^3-8}{x-2},\ x\neq 2$$

For all values of x not equal to 2, you can use standard curve sketching techniques. But the curve is not defined at $x=2$. There is a hole in the graph. So let's get an idea of the behavior of the curve around $x=2$.



Set your calculator to 4 decimal accuracy and complete the chart.

x	1.75	1.9	1.99	1.999	2	2.001	2.01	2.1	2.25
$f(x)$									

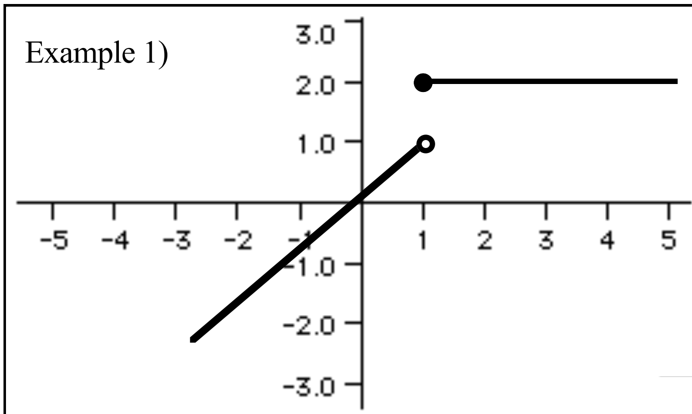
It should be obvious that as *x* gets closer and closer to 2, the value of $f(x)$ becomes closer and closer to ________.

We will say that the **limit** of $f(x)$ as *x* approaches 2 is 12 and this is written as $\lim\limits_{x\to 2} f(x)=12$ or $\lim\limits_{x\to 2}\frac{x^3-8}{x-2}=12$.

The informal definition of a limit is "what is happening to *y* as *x* gets close to a certain number." In order for a limit to exist, we must be approaching the same *y*-value as we approach some value *c* from either the left or the right side. If this does not happen, we say that the limit does not exist (DNE) as we approach *c*.

If we want the limit of $f(x)$ as we approach some value of *c* from the left hand side, we will write $\lim\limits_{x\to c^-} f(x)$.
If we want the limit of $f(x)$ as we approach some value of *c* from the right hand side, we will write $\lim\limits_{x\to c^+} f(x)$.
In order for a limit to exist at *c*, $\lim\limits_{x\to c^-} f(x)$ must equal $\lim\limits_{x\to c^+} f(x)$ and we say $\lim\limits_{x\to c} f(x)=L$.

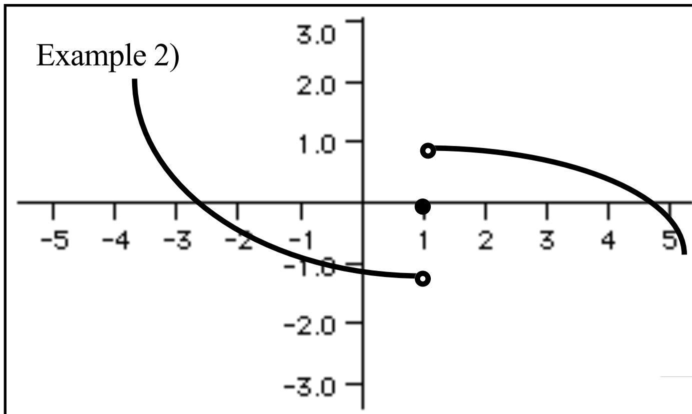
Example 1)



$\lim\limits_{x\to 1^-} f(x)=$ ______ $\lim\limits_{x\to 1^+} f(x)=$ ______

$\lim\limits_{x\to 1} f(x)=$ ______ $f(1)=$ ______

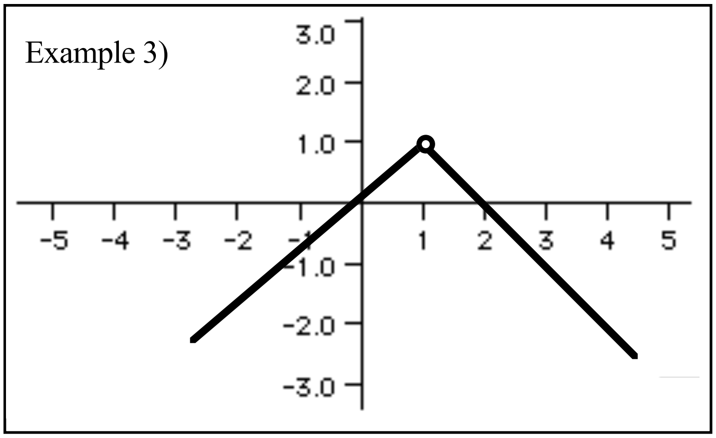
Example 2)



$\lim\limits_{x\to 1^-} f(x)=$ ______ $\lim\limits_{x\to 1^+} f(x)=$ ______

$\lim\limits_{x\to 1} f(x)=$ ______ $f(1)=$ ______

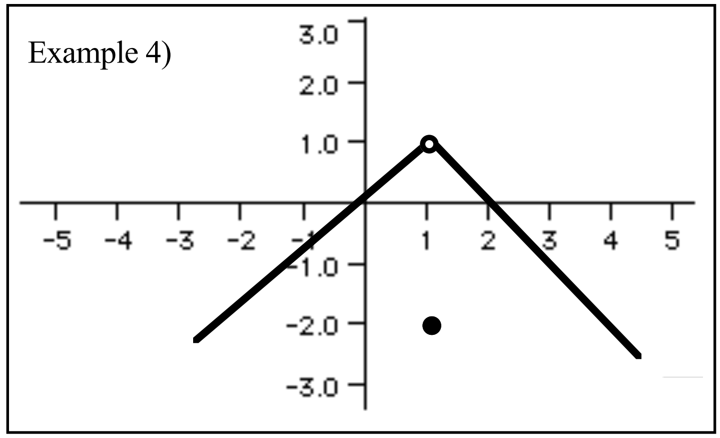
Example 3)



$\lim_{x \to 1^-} f(x) =$ ______ $\lim_{x \to 1^+} f(x) =$ ______

$\lim_{x \to 1} f(x) =$ ______ $f(1) =$ ______

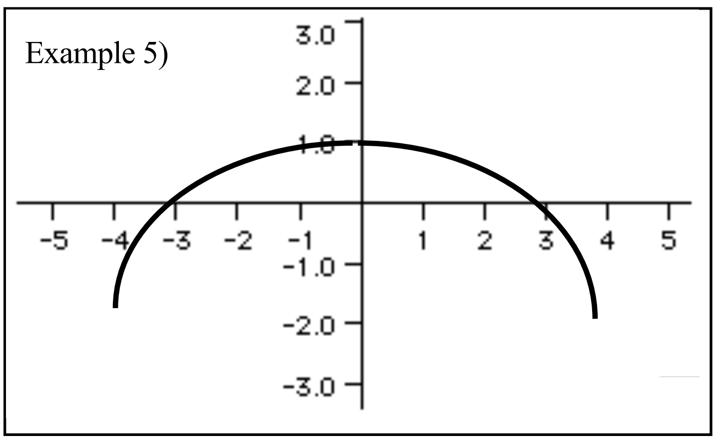
Example 4)



$\lim_{x \to 1^-} f(x) =$ ______ $\lim_{x \to 1^+} f(x) =$ ______

$\lim_{x \to 1} f(x) =$ ______ $f(1) =$ ______

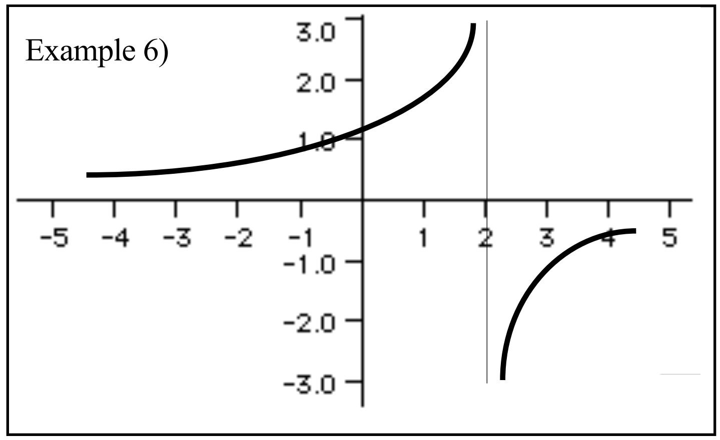
Example 5)



$\lim_{x \to 0^-} f(x) =$ ______ $\lim_{x \to 0^+} f(x) =$ ______

$\lim_{x \to 0} f(x) =$ ______ $f(0) =$ ______

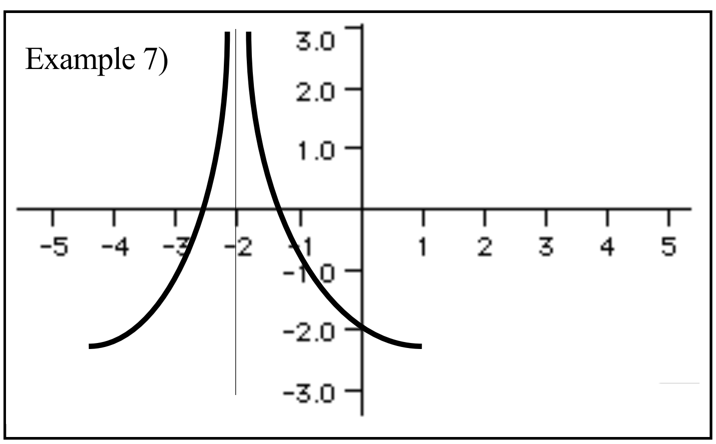
Example 6)



$\lim_{x \to 2^-} f(x) =$ ______ $\lim_{x \to 2^+} f(x) =$ ______

$\lim_{x \to 2} f(x) =$ ______ $f(2) =$ ______

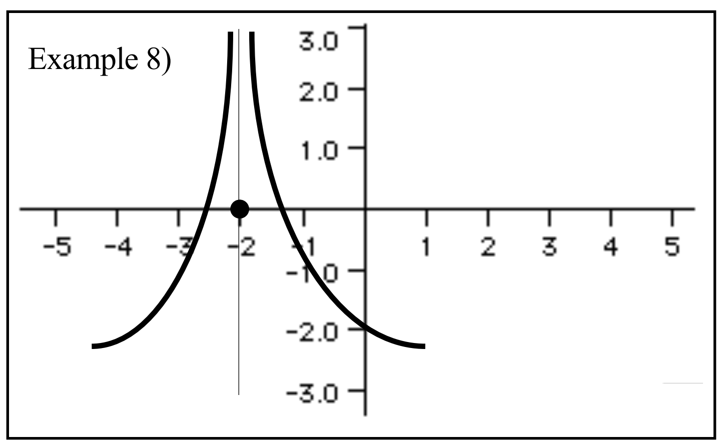
Example 7)



$\lim_{x \to -2^-} f(x) =$ ______ $\lim_{x \to -2^+} f(x) =$ ______

$\lim_{x \to -2} f(x) =$ ______ $f(-2) =$ ______

Example 8)



$\lim_{x \to -2^-} f(x) =$ ______ $\lim_{x \to -2^+} f(x) =$ ______

$\lim_{x \to -2} f(x) =$ ______ $f(-2) =$ ______

The concept of limits as x approaches infinity means the following: "what happens to y as x gets infinitely large." We are interested in what is happening to the y-value as the curve gets farther and farther to the right. We can also talk about limits as x approaches negative infinity. This means what is happening to the y-value as the curve gets farther and farther to the left. The terminology we use are the following: $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

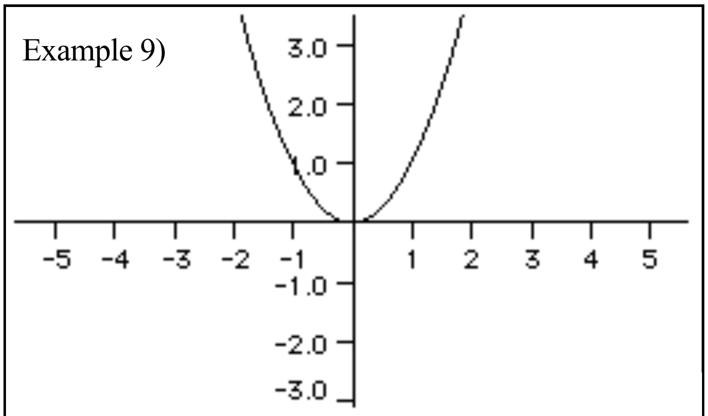
Although we use the term "as x approaches infinity", realize that x cannot approach infinity as infinity does not exist. The term "x approaches infinity" is just a convenient way to talk about the curve infinitely far to the right.

Note that it makes no sense to talk about $\lim_{x\to\infty^+} f(x)$ or $\lim_{x\to-\infty^-} f(x)$. Why? ______________________________

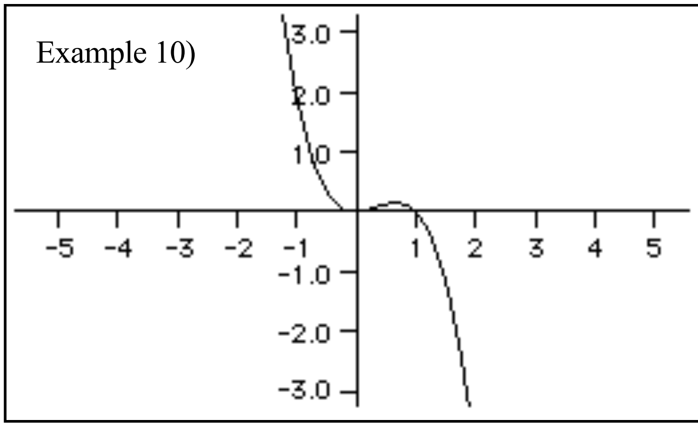
There are only 5 possibilities for $\lim_{x\to\infty} f(x)$ or $\lim_{x\to-\infty} f(x)$:

- the curve can go up forever. In that case, the limit does not exist. For convenience sake, we will say $\lim_{x\to\infty} f(x) = \infty$
- the curve can go down forever. In that case, the limit does not exist. For convenience sake we will say $\lim_{x\to\infty} f(x) = -\infty$

Example 9)



Example 10)

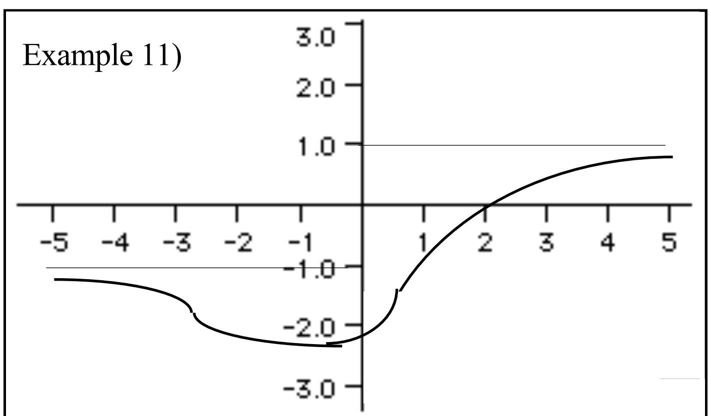


In this case, $\lim_{x\to-\infty} f(x) =$ ______________

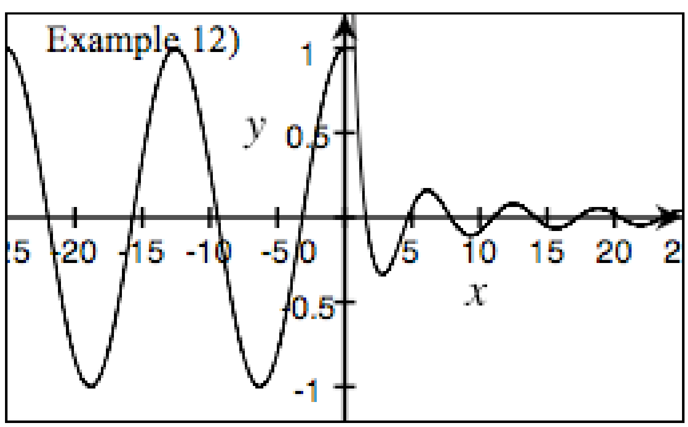
In this case, $\lim_{x\to-\infty} f(x) =$ ______________

- the curve can become asymptotic to a line. In that case the limit as x approaches infinity is a value.
- the curve can level off to a line. In that case, the limit as x approaches infinity is a value. Or the curve can oscillate between two values and the limit does not exist.

Example 11)



Example 12)

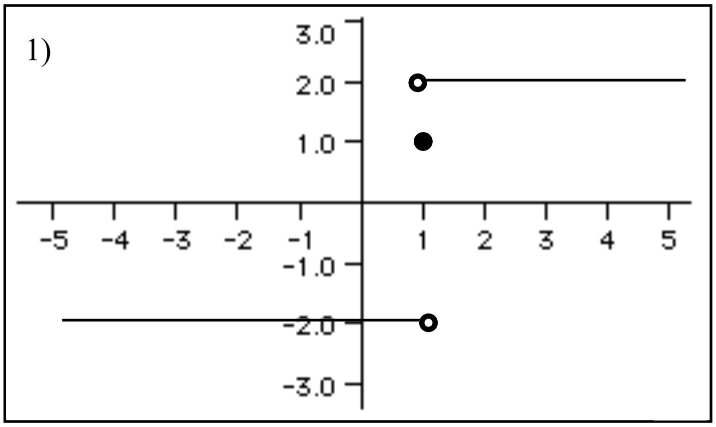


In this case, $\lim_{x\to\infty} f(x) =$ _____ and $\lim_{x\to-\infty} f(x) =$ _____

In this case, $\lim_{x\to\infty} f(x) =$ _____ and $\lim_{x\to-\infty} f(x) =$ _____

Graphical Approach to Limits - Homework

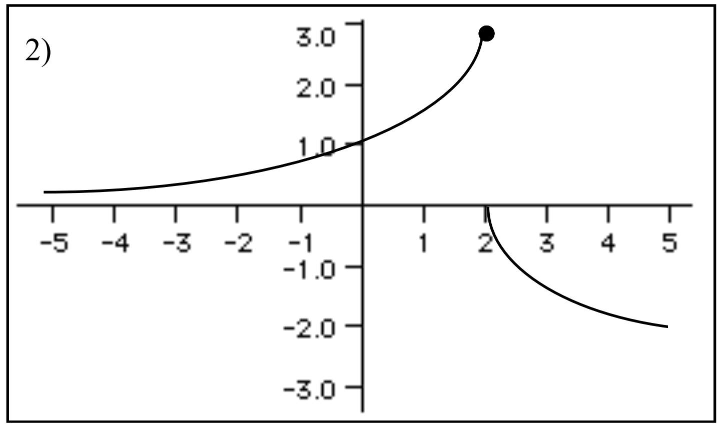
1)



a) $\lim_{x\to1^-} f(x)$ b) $\lim_{x\to1^+} f(x)$ c) $\lim_{x\to1} f(x)$

d) $f(1)$ e) $\lim_{x\to-\infty} f(x)$ f) $\lim_{x\to\infty} f(x)$

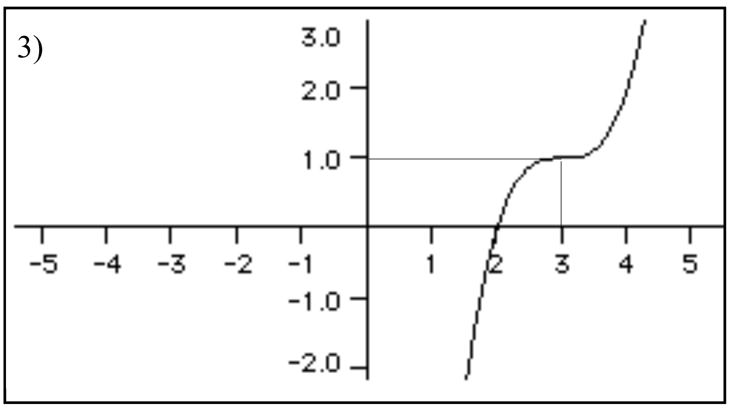
2)



a) $\lim_{x\to2^-} f(x)$ b) $\lim_{x\to2^+} f(x)$ c) $\lim_{x\to2} f(x)$

d) $f(2)$ e) $\lim_{x\to-\infty} f(x)$ f) $\lim_{x\to\infty} f(x)$

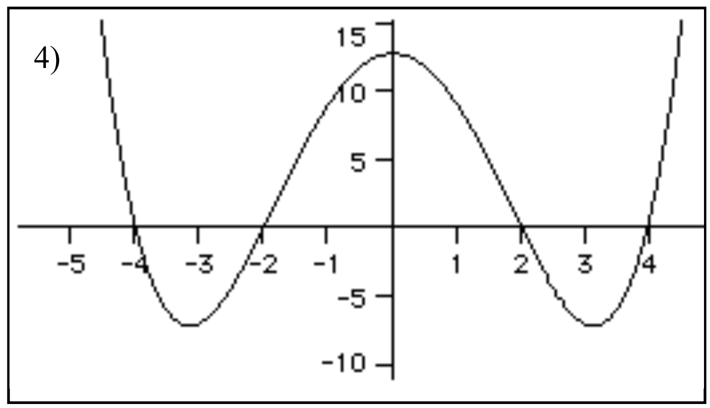
3)



a) $\lim_{x\to3^-} f(x)$ b) $\lim_{x\to3^+} f(x)$ c) $\lim_{x\to3} f(x)$

d) $f(3)$ e) $\lim_{x\to-\infty} f(x)$ f) $\lim_{x\to\infty} f(x)$

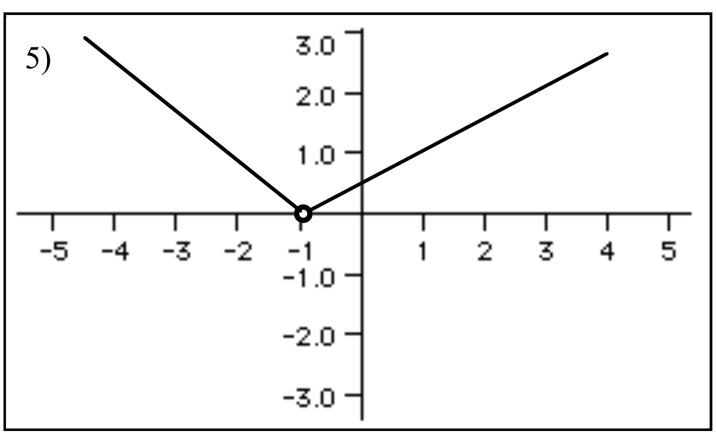
4)



a) $\lim_{x\to0^-} f(x)$ b) $\lim_{x\to0^+} f(x)$ c) $\lim_{x\to0} f(x)$

d) $f(0)$ e) $\lim_{x\to-\infty} f(x)$ f) $\lim_{x\to\infty} f(x)$

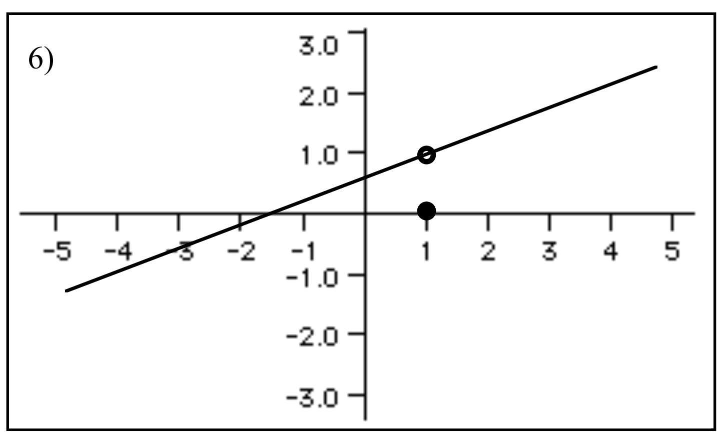
5)



a) $\lim_{x\to-1^-} f(x)$ b) $\lim_{x\to-1^+} f(x)$ c) $\lim_{x\to-1} f(x)$

d) $f(-1)$ e) $\lim_{x\to-\infty} f(x)$ f) $\lim_{x\to\infty} f(x)$

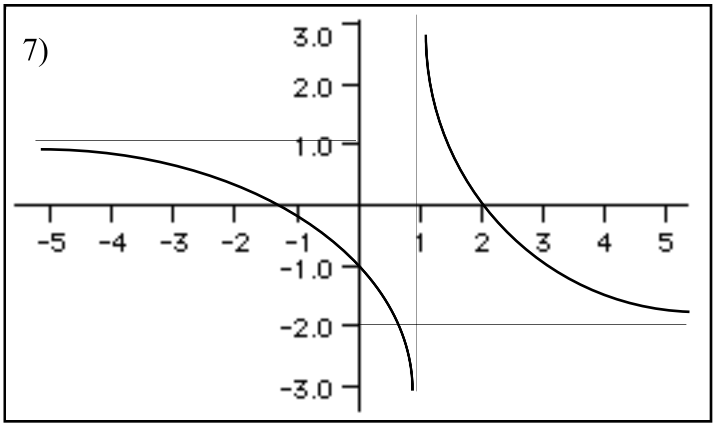
6)



a) $\lim_{x\to-1^-} f(x)$ b) $\lim_{x\to-1^+} f(x)$ c) $\lim_{x\to-1} f(x)$

d) $f(-1)$ e) $\lim_{x\to-\infty} f(x)$ f) $\lim_{x\to\infty} f(x)$

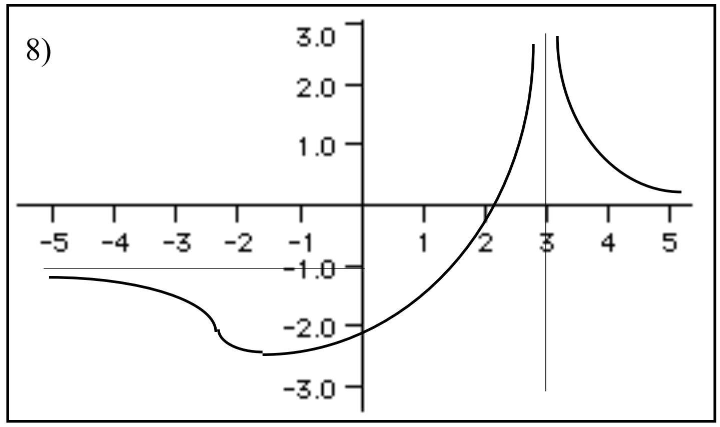
7)



a) $\lim\limits_{x\to1^-} f(x)$ b) $\lim\limits_{x\to1^+} f(x)$ c) $\lim\limits_{x\to1} f(x)$

d) $f(1)$ e) $\lim\limits_{x\to-\infty} f(x)$ f) $\lim\limits_{x\to\infty} f(x)$

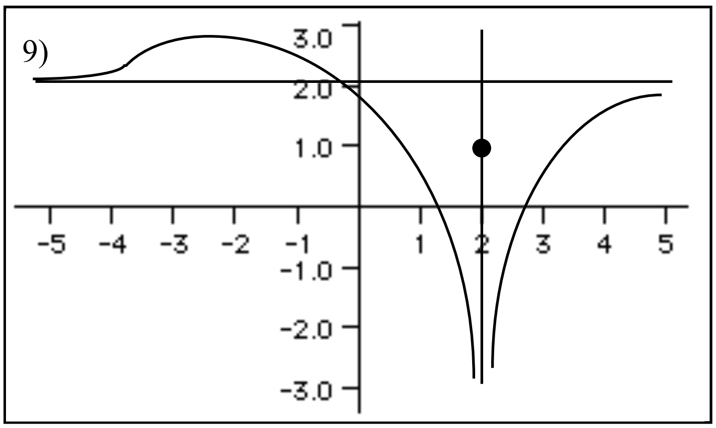
8)



a) $\lim\limits_{x\to3^-} f(x)$ b) $\lim\limits_{x\to3^+} f(x)$ c) $\lim\limits_{x\to3} f(x)$

d) $f(3)$ e) $\lim\limits_{x\to-\infty} f(x)$ f) $\lim\limits_{x\to\infty} f(x)$

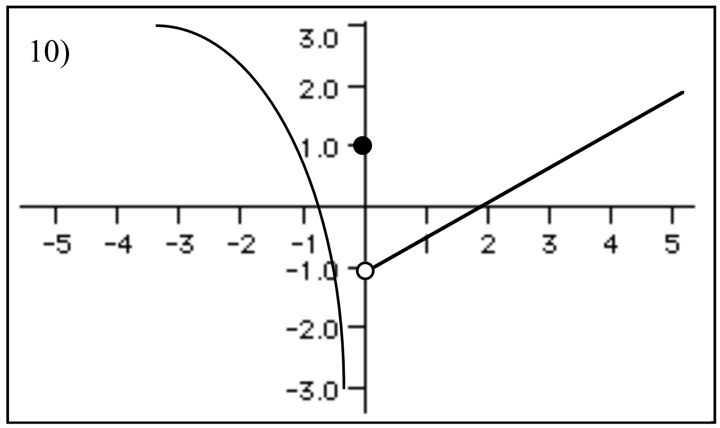
9)



a) $\lim\limits_{x\to2^-} f(x)$ b) $\lim\limits_{x\to2^+} f(x)$ c) $\lim\limits_{x\to2} f(x)$

d) $f(2)$ e) $\lim\limits_{x\to-\infty} f(x)$ f) $\lim\limits_{x\to\infty} f(x)$

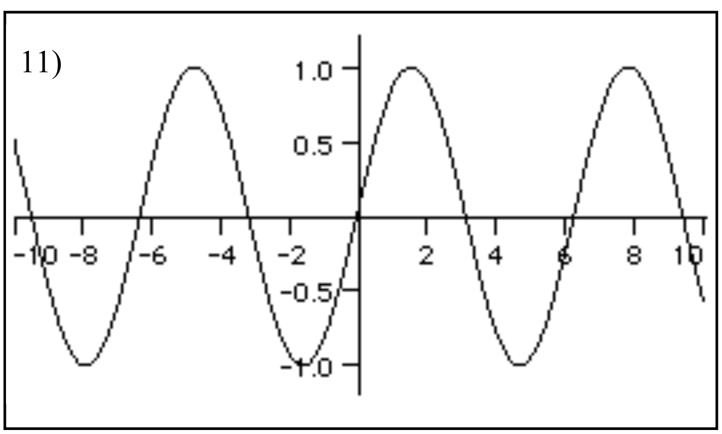
10)



a) $\lim\limits_{x\to0^-} f(x)$ b) $\lim\limits_{x\to0^+} f(x)$ c) $\lim\limits_{x\to0} f(x)$

d) $f(0)$ e) $\lim\limits_{x\to-\infty} f(x)$ f) $\lim\limits_{x\to\infty} f(x)$

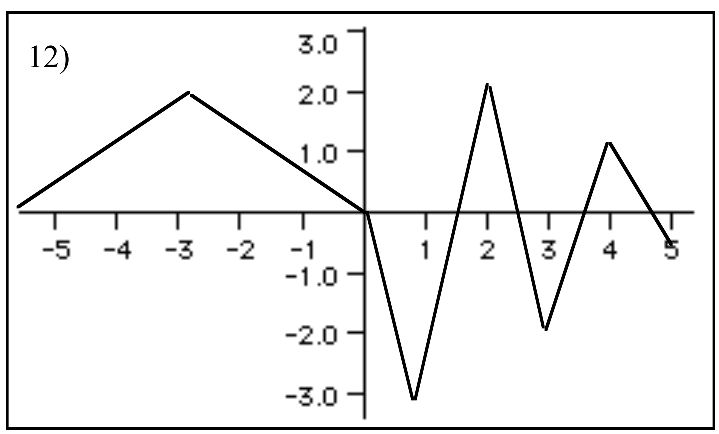
11)



a) $\lim\limits_{x\to0^-} f(x)$ b) $\lim\limits_{x\to0^+} f(x)$ c) $\lim\limits_{x\to0} f(x)$

d) $f(0)$ e) $\lim\limits_{x\to-\infty} f(x)$ f) $\lim\limits_{x\to\infty} f(x)$

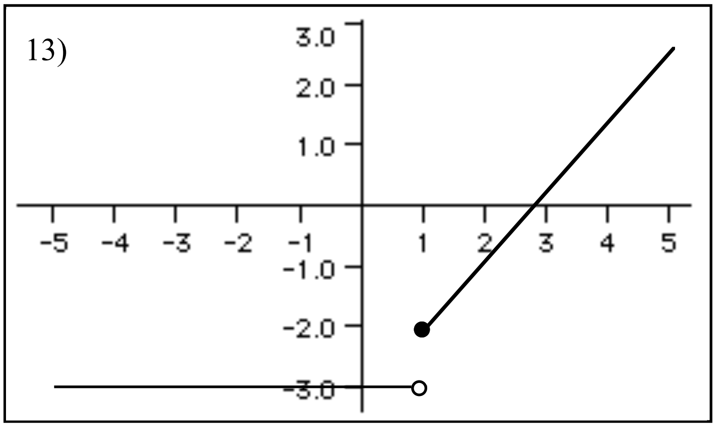
12)



a) $\lim\limits_{x\to0^-} f(x)$ b) $\lim\limits_{x\to0^+} f(x)$ c) $\lim\limits_{x\to0} f(x)$

d) $f(0)$ e) $\lim\limits_{x\to-\infty} f(x)$ f) $\lim\limits_{x\to\infty} f(x)$

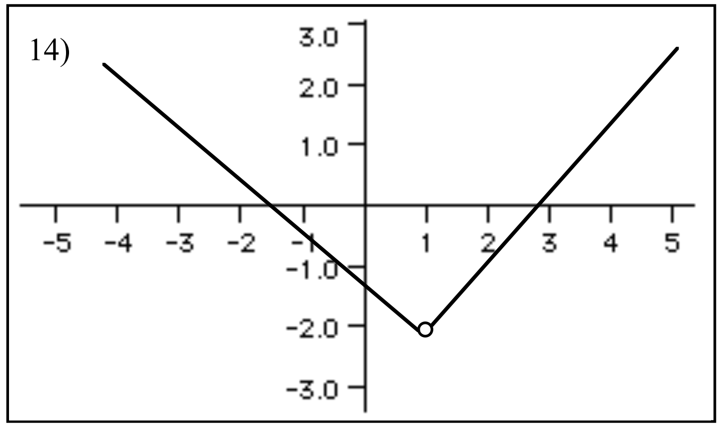
13)



a) $\lim_{x\to 1^-} f(x)$ b) $\lim_{x\to 1^+} f(x)$ c) $\lim_{x\to 1} f(x)$

d) $f(1)$ e) $\lim_{x\to -\infty} f(x)$ f) $\lim_{x\to \infty} f(x)$

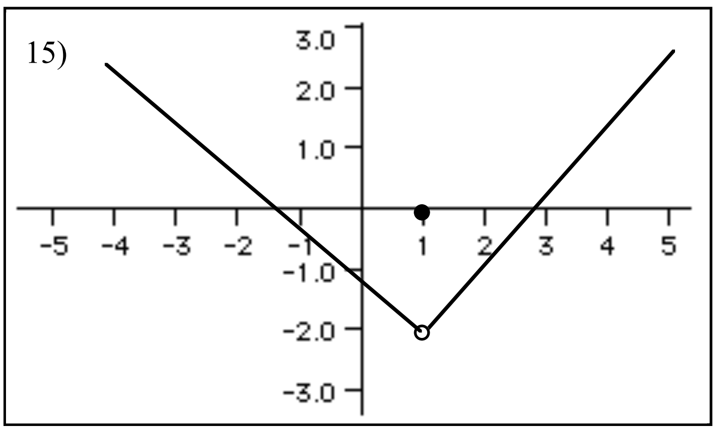
14)



a) $\lim_{x\to 1^-} f(x)$ b) $\lim_{x\to 1^+} f(x)$ c) $\lim_{x\to 1} f(x)$

d) $f(1)$ e) $\lim_{x\to -\infty} f(x)$ f) $\lim_{x\to \infty} f(x)$

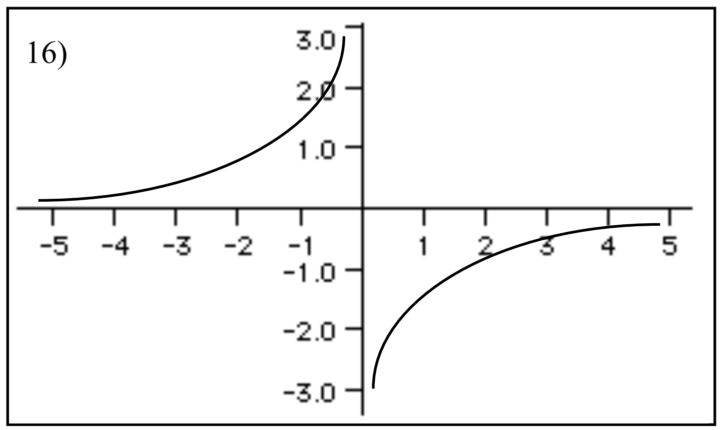
15)



a) $\lim_{x\to 1^-} f(x)$ b) $\lim_{x\to 1^+} f(x)$ c) $\lim_{x\to 1} f(x)$

d) $f(1)$ e) $\lim_{x\to -\infty} f(x)$ f) $\lim_{x\to \infty} f(x)$

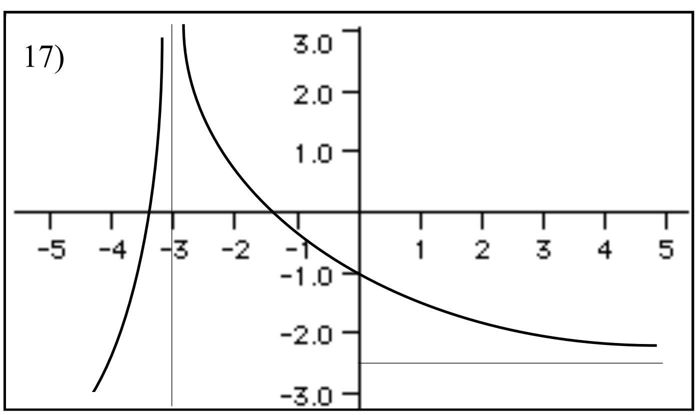
16)



a) $\lim_{x\to 0^-} f(x)$ b) $\lim_{x\to 0^+} f(x)$ c) $\lim_{x\to 0} f(x)$

d) $f(0)$ e) $\lim_{x\to -\infty} f(x)$ f) $\lim_{x\to \infty} f(x)$

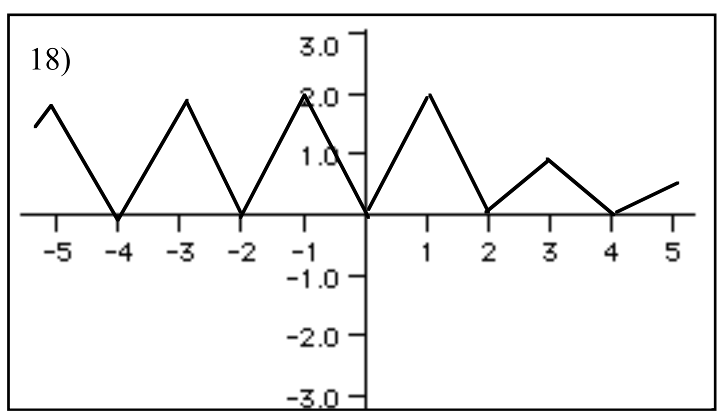
17)



a) $\lim_{x\to -3^-} f(x)$ b) $\lim_{x\to -3^+} f(x)$ c) $\lim_{x\to -3} f(x)$

d) $f(-3)$ e) $\lim_{x\to -\infty} f(x)$ f) $\lim_{x\to \infty} f(x)$

18)



a) $\lim_{x\to 0^-} f(x)$ b) $\lim_{x\to 0^+} f(x)$ c) $\lim_{x\to 0} f(x)$

d) $f(0)$ e) $\lim_{x\to -\infty} f(x)$ f) $\lim_{x\to \infty} f(x)$

Finding Limits Algebraically - Classwork

We are going to now determine limits without benefit of looking at a graph, that is $\lim_{x \to a} f(x)$.
There are three steps to remember: 1) plug in *a*
2) Factor/cancel and go back to step 1
3) ∞, -∞, or DNE

Example 1) find $\lim_{x \to -2} x^2 - 4x + 1$

You can do this by plugging in.

Example 2) find $\lim_{x \to -2} \dfrac{2x-6}{x-2}$

You can also do this by plugging in.

Example 3) find $\lim_{x \to -2} \dfrac{x^2-2x-8}{x^2-4}$

Plug in and you get $\dfrac{0}{0}$ - no good

So attempt to factor and cancel

Example 4) find $\lim_{x \to 1} \dfrac{x^2-2x+1}{x^3-1}$

Plug in and you get $\dfrac{0}{0}$ - no good

So attempt to factor and cancel

If steps 1 and 2 do not work (you have a zero in the denominator, your answer is one of the following:

∞	**-∞**	**Does Not Exist (DNE)**

To determine which, you must split your limit into two separate limits.: $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a+} f(x)$. Make a sign chart by plugging in a number close to *a* on the left side and determining its sign. You will also plug in a number close to *a* on the right side and determine its sign. **Each of these will be some form of ∞, either positive or negative.** Only if they are the same will the limit be ∞ or -∞.

What this says is that in this case, $\lim_{x \to a^-} f(x) =$ some form of ∞ and $\lim_{x \to a+} f(x) =$ some form of ∞
You need to check whether they are the same.

Example 5) find $\lim_{x \to 2} \dfrac{2x+5}{x-2}$

Step 1) Plug in - $\dfrac{9}{0}$ - no good Step 2) - No factoring/cancel So your answer is ∞, -∞ or DNE

Example 6) find $\lim_{x\to 0}\frac{4}{x^2}$

Step 1) Plug in - $\frac{4}{0}$ - no good Step 2) - No factoring/cancel So your answer is ∞, $-\infty$ or DNE

Example 7) find $\lim_{x\to -3}\frac{x^2+2x-3}{x^2+6x+9}$

Example 8) find $\lim_{x\to 2}\frac{2x-4}{x^3-6x^2+12x-8}$

Example 9) $f(x)=\begin{cases} x^2-4, x\geq 1 \\ -2x-1, x<1 \end{cases}$ find $\lim_{x\to 1} f(x)$

Example 10) $f(x)=\begin{cases} \frac{x}{x-2}, x\geq 2 \\ \frac{x-3}{x-2}, x<2 \end{cases}$ find $\lim_{x\to 2} f(x)$

Example 11) $\lim_{x\to 0}\frac{\sqrt{x+2}-\sqrt{2}}{x}$

Finally, we are interested also in problems of the type: $\lim_{x\to\pm\infty} f(x)$. Here are the rules:

Write $f(x)$ as a fraction. 1) If the highest power of x appears in the denominator (bottom heavy), $\lim_{x\to\pm\infty} f(x)=0$

2) If the highest power of x appears in the numerator (top heavy), $\lim_{x\to\pm\infty} f(x)=\pm\infty$ plug in very large or small numbers and determine the sign of the answer

3) If the highest power of x appears both in the numerator and denominator (powers equal), $\lim_{x\to\pm\infty} f(x)=\frac{\text{coefficient of numerator's highest power}}{\text{coefficient of denominator's highest power}}$

Example 12) $\lim_{x\to\infty}\frac{4x^2+50}{x^3-85}$

Example 13) $\lim_{x\to\infty}\frac{4x^3-5x^2+3x-1}{5x^3-7x-25}$

Example 14) $\lim_{x\to\infty}\frac{3x^3-23}{4x-1}$

Example 15) $\lim_{x\to -\infty}\frac{4x-5x^2+3}{\frac{1}{x}}$

Example 16) $\lim_{x\to\infty}\frac{\sqrt{x^2-3x}}{2x+1}$

Example 17) $\lim_{x\to -\infty}\frac{\sqrt{x^2-3x}}{2x+1}$

Finding Limits Algebraically - Homework

1) $\lim_{x \to 5} 12$

2) $\lim_{x \to 0} \pi$

3) $\lim_{x \to 2} 4x$

4) $\lim_{x \to 5} 3x^2 - 4x - 1$

5) $\lim_{x \to 0^-} 5x^3 - 7x^2 + 2^x - 2$

6) $\lim_{y \to -1} 3y^4 - 6y^3 - 2y$

7) $\lim_{x \to 4} \frac{2x-4}{x-1}$

8) $\lim_{x \to -2} \frac{x^2+4x+4}{x^2}$

9) $\lim_{x \to 1} \frac{2x-2}{x-1}$

10) $\lim_{x \to 4} \frac{x^2-16}{x-4}$

11) $\lim_{t \to -2} \frac{t^3+8}{t+2}$

12) $\lim_{x \to 2} \frac{x^2-4x+4}{x^2+x-6}$

13) $\lim_{x \to -1} \frac{x^2+6x+5}{x^2-3x-4}$

14) $\lim_{x \to 1} \frac{x^3+x^2-5x+3}{x^3-3x+2}$

15) $\lim_{x \to 3} \frac{x}{x-3}$

16) $\lim_{x \to 5} \frac{x}{x^2-25}$

17) $\lim_{y \to 6} \frac{y+6}{y^2-36}$

18) $\lim_{x \to 4} \frac{3-x}{x^2-2x-8}$

19) $\lim_{x \to 1} \frac{4}{x^2-2x+1}$

20) $\lim_{x \to 5} \frac{x}{|x-5|}$

21) $\lim_{x \to 3} \frac{-x^2}{x^2-6x+9}$

22) $f(x)=\begin{cases} x-1, x\ge 3 \\ 2x-3, x<3 \end{cases}$ find $\lim\limits_{x\to 3} f(x)$

23) $f(x)=\begin{cases} x^3-1, x\ge -1 \\ 2x, x<-1 \end{cases}$ find $\lim\limits_{x\to -1} f(x)$

24) $f(x)=\begin{cases} \dfrac{x-2}{x-1}, x\ge 1 \\ \dfrac{x}{x-1}, x<1 \end{cases}$ find $\lim\limits_{x\to 1} f(x)$

25) $\lim\limits_{x\to 0}\dfrac{\sqrt{x+4}-2}{x}$

26) Let $f(x)=\begin{cases} x^2-2x-3, x\ne 2 \\ k-3, x=2 \end{cases}$

find k such that $\lim\limits_{x\to 2} f(x)=f(2)$

27) $f(x)=\begin{cases} \dfrac{x^2-49}{x-7}, x\ne 7 \\ k^2-2, x=7 \end{cases}$

find k such that $\lim\limits_{x\to 7} f(x)=f(7)$

28) $\lim\limits_{x\to\infty} 6$

29) $\lim\limits_{x\to\infty}(-2x+11)$

30) $\lim\limits_{x\to-\infty}\left(3x^4-3x^3+5x^2+8x-3\right)$

31) $\lim\limits_{x\to\infty}\dfrac{2x-3}{4x+5}$

32) $\lim\limits_{x\to-\infty}\dfrac{7-3x^3}{2x^3+1}$

33) $\lim\limits_{x\to\infty}\dfrac{2}{5x-3}$

34) $\lim\limits_{x\to-\infty}\dfrac{2x+30}{6x^{12}-5}$

35) $\lim\limits_{x\to\infty}\dfrac{4x^4}{6x^3-19}$

36) $\lim\limits_{x\to-\infty}\dfrac{4x^2-3x-2-5x^3}{9x^2+9x+7}$

37) $\lim\limits_{x\to\infty}\dfrac{x}{\sqrt{x^2+4}}$

38) $\lim\limits_{x\to-\infty}\dfrac{x}{\sqrt{x^2+4}}$

39) $\lim\limits_{x\to-\infty}\dfrac{\sqrt{3x^2+x}}{x^2-1}$

Definition of Derivative - Classwork

Let us do a bit of reviewing. For the function $f(x) = x^2 - 3x + 1$, find

a) the slope of the tangent line at $x = 2$

b) the slope of the tangent line at $x = 0$.

c) the slope of the tangent line at $x = -1$.

Obviously we have duplicated our efforts a great deal. The process is the same - only the point at which we find the slope of the tangent changes. With that in mind, we are ready to introduce a basic concept of calculus.

The **derivative** of a function is a formula for the slope of the tangent line to that function at any point x. The process of taking derivatives is called **differentiation**.

We now define the derivative of a function $f(x)$ as $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$. This mimics the procedure used above but it calculates the slope of the tangent line at any generic point x. Notice that we can now talk about this process in terms of a limit. When you do your cancellation, you are essentially performing the limit procedures we did in the last section.

Example 1) For the function $f(x) = x^2 - 3x + 1$, find its derivative and evaluate the derivative at $x = 2$, 0, and -1.

We now need to specify some notation for the derivative. When the function is defined as $f(x)$, the derivative will be written as $f'(x)$ or f'. When the function is written in the form of $y =$, the derivative is written as y' or $\dfrac{dy}{dx}$. The latter looks like a fraction but for now will be one entity, the derivative of y and it is pronounced "dy dx."

Example 2) $f(x) = 4x$, find $f'(x)$

Example 3) $f(x) = x^2 + x$, find $f'(x)$

Example 4) $f(x) = 2x^2 - 5x + 6$, find $f'(x)$ and $f'(3)$

Example 5) $y = \dfrac{4}{x}$, find $\dfrac{dy}{dx}$

For the following functions, find their derivative and evaluate at $x = 2$, -4, 0 and π. Use proper notation.

1. $y = 2x$

Answers: $y' = 2$

$y'(2) = 2 \quad y'(-4) = 2 \quad y'(0) = 2 \quad y'(\pi) = 2$

2. $y = x^2 - 5$

Answers: $y' = 2x$

$y'(2) = 4 \quad y'(-4) = -8 \quad y'(0) = 0 \quad y'(\pi) = 2\pi$

3. $f(x) = x^2 + 3x - 4$

Answers: $f'(x) = 2x + 3$

$y'(2) = 7 \quad y'(-4) = -5 \quad y'(0) = 3 \quad y'(\pi) = 2\pi + 3$

4. $f(x) = 4x^2 - 6x + 1$

Answers: $f'(x) = 8x - 6$

$y'(2) = 10 \quad y'(-4) = -38 \quad y'(0) = -6 \quad y'(\pi) = 8\pi - 6$

5. $f(x) = x^3 + 2x$

Answers: $f'(x) = 3x^2 + 2$

$y'(2) = 14 \quad y'(-4) = 50 \quad y'(0) = 2 \quad y'(\pi) = 3\pi^2 + 2$

6. $f(x) = \dfrac{5}{x} + 1$

Answers: $f'(x) = \dfrac{-5}{x^2}$

$y'(2) = \dfrac{-5}{4} \quad y'(-4) = \dfrac{-5}{16} \quad y'(0) = DNE \quad y'(\pi) = \dfrac{-5}{\pi^2}$

7. $f(x) = \dfrac{-1}{x^2}$

Answers: $f'(x) = \dfrac{2}{x^3}$

$y'(2) = \dfrac{1}{4} \quad y'(-4) = \dfrac{-1}{32} \quad y'(0) = DNE \quad y'(\pi) = \dfrac{2}{\pi^3}$

8. $f(x) = \sqrt{x}$

Answers: $f'(x) = \dfrac{1}{2\sqrt{x}}$

$y'(2) = \dfrac{1}{2\sqrt{2}} \quad y'(-4) = DNE \quad y'(0) = DNE \quad y'(\pi) = \dfrac{1}{2\sqrt{\pi}}$

Derivatives Using Technology

While the TI 84, cannot find derivatives, it can come very close to finding the value of the derivative at a specified point. Only calculators like the TI-89 with CAS (Computer Algebra System) can find the actual derivative of a function because of its ability to work symbolically.

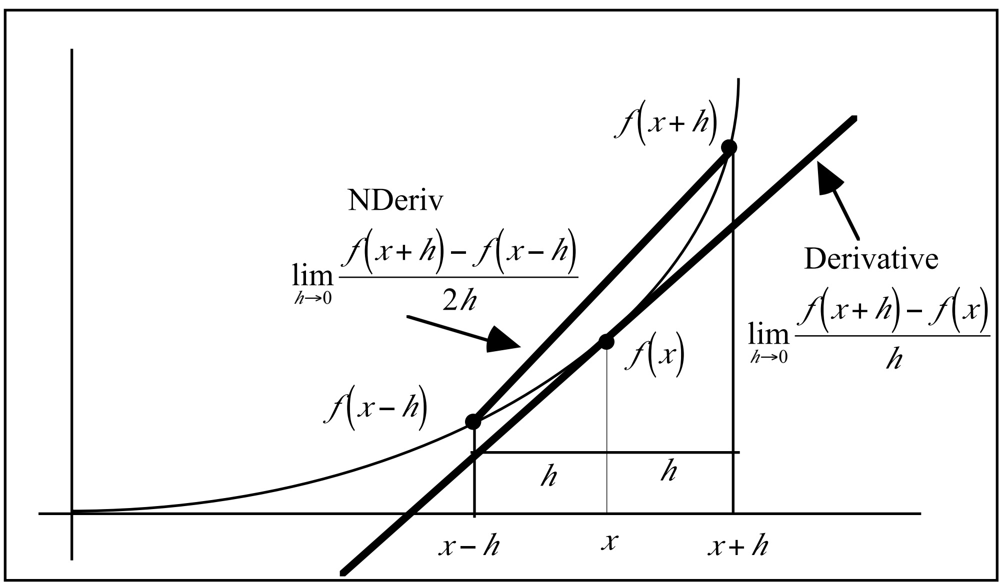
The derivative is defined as the slope of a tangent line to a curve $f(x)$. Its formal definition is this:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The TI calculators use another function to approximate the slope of a tangent line to a curve $f(x)$ at a point. It is called the **NUMERICAL DERIVATIVE**. The TI 83, 83+ , and 84 abbreviates this function as **nDeriv**. The definition of the numerical derivative is as follows:

$$\textbf{nDeriv} = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}$$

This function calculates the slope of the secant line between 2 points very close to x. The variable h is the distance between x and either point. The closer the 2 points are to x, the closer **nDeriv** will be to the actual derivative.*



To use nDeriv with the TI-84, 83+

1) Place the functions you wish to use in Y_1
2) **nDeriv** is MATH 8. Use **nDeriv**(Y_1,X,the x coordinate of the point where you wish to calculate the derivative)

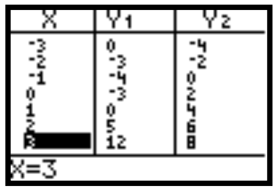
TI 84 Example

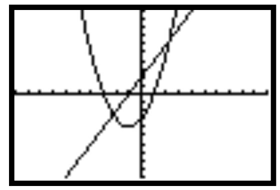
1) $Y_1 = X^2$
2) **nDeriv**(Y-VARS Function Y_1,X,1)

The answer should be 2.

It is not necessary to use an h with **nDeriv**. The calculator assumes the value .001 as h. You can specify a smaller value of h using **nDeriv**(Y_1,X,X value, h).

Setting up the calculator with a function such as $f(x) = x^2 + 2x - 3$ and a numerical derivative of the function leads to an interesting result. The TI-84 statement Y_2 = **nDeriv**(Y_1,X,X) says to take the numerical derivative of the function Y_1 with respect to *x* at each value of *x* that is graphed. The resulting TI-84 table shows each point of the function (*x*,Y_1) and Y_2 is the slope of the tangent line at that point. For instance, $f(x)$ goes through the point (-3, 0) and the slope of the tangent line at that point is -4. It goes through the point (0,-3) and the slope of the tangent line is 2. If you graph all points (X,Y_2) you get a straight line. This straight line is the line $y = 2x + 2$, the result of taking the derivative of the function $x^2 + 2x - 3$. This line is **not** tangent to $x^2 + 2x - 3$. The *y* values are the slopes of the tangent line to $f(x)$ at each point *x*.





* There are times when the **nDeriv** function will give a false answer. In general, these are places where the derivative of a function is not defined. The best example of this is $y = |x|$. We will see that the derivative of $y = |x|$ is not defined at $x = 0$ (why?). And yet, if you press in **nDeriv**(Y_1,X,0) on the TI 84 or by the method described using the other calculators, you will get the value zero. Why does it do so? Look at the definition of **nDeriv**.

<u>What does the calculator's ability to find derivatives at points mean to A.P. students?</u>

First of all, with the calculator, you are not finding derivatives symbolically, only numerically. Since a derivative is a **formula** for the slope of a tangent line to a curve $y = f(x)$, we must learn the techniques for finding these formulas. Much of the first semester will be devoted to that need and the calculators really will not be of much use.

But, when you are asked to find the slope of the tangent line to a curve at a single point, the calculator will serve well in two ways. First, if on exams, including the A.P. exam, you are asked to find the derivative at a point, you will still have to do the work symbolically, but the calculator will serve as an answer book for you. Second, if the expression that you need to find the slope of the tangent line for is very complicated, you could simply do it numerically on the calculator, ignoring the techniques you will have learned. On the "calculator necessary" section of the A.P. exam, this is a skill that will serve you well.

In general, the calculator serves as an aide to *verify* what we find by calculus techniques. If you decide to rely solely on the calculator, you are making a very bad decision and you will have absolutely no shot on the A.P. exam.

Techniques of Differentiation - Classwork

Taking derivatives is a a process that is vital in calculus. In order to take derivatives, there are rules that will make the process simpler than having to use the definition of the derivative.

1. The constant rule: The derivative of a constant function is 0. That is, if c is a real number, then $\frac{d}{dx}[c]=0$.

a) $y=7$ $\quad y'=$

b) $f(x)=0$ $\quad f'(x)=$

c) $s(t)=-8$ $\quad s'(t)=$

d) $y=a\pi^3$ $\quad \frac{dy}{dx}=$

2. The single variable rule: The derivative of x is 1. $\frac{d}{dx}[x]=1$. This is consistent with the fact that the slope of the line $y=x$ is 1.

a) $y=x$ $\quad y'=$

b) $f(x)=x$ $\quad f'(x)=$

c) $s(t)=t$ $\quad s'(t)=$

3. The power rule: If n is a rational number then the function x^n is differentiable and $\frac{d}{dx}\left[x^n\right]=nx^{n-1}$.

Take the derivatives of the following. Use correct notation.

a) $y=x^2$

b) $f(x)=x^6$

c) $s(t)=t^{30}$

d) $y=\sqrt{x}$

e) $y=\frac{1}{x}$

f) $f(x)=\frac{1}{x^3}$

g) $s(t)=\frac{1}{\sqrt[3]{t}}$

h) $y=\frac{1}{x^{3/4}}$

4) The constant multiple rule: If f is a differentiable function and c is a real number, then $\frac{d}{dx}[cf(x)]=cf'(x)$

Take the derivatives of the following. Use correct notation.

a) $y=\frac{2}{x^2}$

b) $f(x)=\frac{4x^3}{3}$

c) $s(t)=-t^5$

d) $y=4\sqrt{x}$

e) $y=\frac{-5}{3x^3}$

f) $f(x)=\frac{-5}{(3x)^3}$

g) $s(t)=\frac{4}{\sqrt{t}}$

h) $y=\frac{-12}{\sqrt[3]{x^5}}$

5. The sum and difference rules. The derivative of a sum or difference is the sum or difference of the derivatives.

$$\frac{d}{dx}\left[f(x)+g(x)\right]=f'(x)+g'(x) \quad \text{and} \quad \frac{d}{dx}\left[f(x)-g(x)\right]=f'(x)-g'(x)$$

Take the derivatives of the following. Use correct notation.

a) $y=x^2+5x-3$

b) $f(x)=x^4-\frac{3}{2}x^3+2x^2+x-6$

c) $y=\frac{4}{x}-\frac{4}{x^2}+\frac{4}{x^3}$

d) $y=6\sqrt{x}\left(\sqrt[3]{x}-2x+6\right)$

e) $f(x)=(2x-3)^2$

f) $y=\left(x^2-x+1\right)^2$

6. <u>The Product Rule:</u> The derivative of the product of two functions is the first times the derivative of the second plus the second times the derivative of the first. $\frac{d}{dx}\left[f(x)\cdot g(x)\right]=f(x)\cdot g'(x)+g(x)\cdot f'(x)$

a) Find y' if $y=\left(4x-2x^2\right)(3x-5)$

without product rule

with product rule

b) Find y' if $y=\left(x^2-x+1\right)^2$

c) Find $f'(x)$ if $f(x)=\left(3x^2-2x+5\right)\left(-5x^4+2x^3-7x^2+x+2\right)$

7. <u>The Quotient Rule</u>: The derivative of the quotient of two functions *f* and *g* can be found using the following:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\cdot f'(x)-f(x)\cdot g'(x)}{\left[g(x)\right]^2}$$

a) Find $\frac{d}{dx}\left[\frac{5x+2}{x^2-1}\right]$

b) Find $\frac{d}{dx}\left[\frac{5x+3}{x^2+4x-2}\right]$

c) Find $\frac{d}{dx}\left[\frac{x^3-4x^2+4x-2}{2x}\right]$

d) Find $\frac{d}{dx}\left[\frac{2x}{x^3-4x^2+4x-2}\right]$

Find an equation of the tangent line to the graph of f at the indicated point and then use your calculator to confirm the results.

a) $y = (x^2 - 4x + 2)(4x - 1)$ at $(1, -3)$

b) $y = \dfrac{x-4}{x^2+3}$ at $\left(2, \dfrac{-2}{7}\right)$

Determine the points at which the graph of the following function has a horizontal tangent.

a) $y = \dfrac{8}{3}x^3 + 5x^2 - 3x - 1$

b) $y = \dfrac{x^2 - 3}{x^2 + 1}$

c) $y = \dfrac{x-1}{x^2+3}$

Use the chart to find $f'(3)$

$g(3)$	$g'(3)$	$h(3)$	$h'(3)$
4	-2	3	π

a) $f(x) = 4g(x) - \dfrac{1}{2}h(x) + 1$

b) $f(x) = g(x)h(x)$

c) $f(x) = \dfrac{g(x)}{2h(x)}$

d) $f(x) = \dfrac{g(x) - h(x)}{g(x)}$

The 2nd derivative of a function $y = f(x)$ can be written as $\dfrac{d^2y}{dx^2}$ or $f''(x)$. The 3rd derivative is $\dfrac{d^3y}{dx^3}$ or $f'''(x)$.
The 2nd derivative of a function is the derivative of the derivative of the function.
For each of the following, find $f''(x)$.

a) $f(x) = \dfrac{8}{3}x^3 - 5x^2 - 7x - 1$

b) $f(x) = \dfrac{x^2 + 4x - 2}{x}$

c) $f(x) = \dfrac{x}{x+1}$

d) $f(x) = 4\sqrt{x} - \dfrac{2}{\sqrt{x}}$

Techniques of Differentiation - Homework

For the following functions, find $f'(x)$ and $f'(c)$ at the indicated value of *c*.

1) $f(x) = x^2 - 6x + 1 \quad c = 0$

2) $f(x) = \frac{1}{x} - \frac{3}{x^2} + \frac{4}{x^3} \quad c = 1$

3) $f(x) = 3\sqrt{x} - \frac{1}{\sqrt[3]{x}} \quad c = 1$

For the following functions, find the derivative using the power rule.

4) $y = \frac{8}{3x^2}$

5) $y = \frac{-9}{\left(3x^2\right)^3}$

6) $y = \frac{6x^{3/2}}{x}$

7) $y = \frac{4x^2 - 5x + 6}{3}$

8) $y = \frac{x^2 - 6x + 2}{2x}$

9) $y = \frac{x^3 + 8}{x + 2}$

10) $y = x^4 - \frac{3}{2}x^3 + 5x^2 - 6x - 2$

11) $y = \frac{x^3 - 3x^2 + 10x - 5}{x^2}$

12) $y = \left(x^2 + 4x\right)(2x - 1)$

13) $y = (x - 2)^3$

14) $y = \sqrt[3]{x} - \sqrt[3]{x^2}$

15) $y = \frac{\left(x^2 - x + 2\right)^2}{x}$

For the following functions, find the derivatives.

16) $y = \left(x^2 - 4x - 6\right)\left(x^3 - 5x^2 - 3x\right)$

17) $y = \frac{3x - 2}{2x + 3}$

18) $y = \frac{x^2 - 4x - 2}{x^2 - 1}$

19) $y = \dfrac{x-1}{\sqrt{x}}$

20) $y = \dfrac{x^2 - x + 1}{\sqrt[3]{x}}$

21) $y = \left(\dfrac{x-3}{x+4}\right)(3x-2)$

22) $y = \dfrac{x-1}{x^2 + 2x + 2}$

23) $y = \dfrac{x^2 + k^2}{x^2 - k^2}$, k is a constant

24) $y = \dfrac{x^2 - k^2}{x^2 + k^2}$, k a constant

Find an equation of the tangent line to the graph of f at the indicated point and then use your calculator to confirm the results.

25) $f(x) = \dfrac{x^2}{x-1}$ at $(2,4)$

26) $f(x) = (x-2)(x^2 - 3x - 1)$ at $(-1,-9)$

27) $f(x) = \dfrac{x^2 - 4x + 2}{2x - 1}$ at $\left(2, -\dfrac{2}{3}\right)$

28) $y = \left(\dfrac{x+3}{x+1}\right)(4x+1)$ at $\left(-\dfrac{1}{2}, -5\right)$

Determine the point(s) at which the graph of the following function has a horizontal tangent.

29) $f(x) = \dfrac{x^2}{x^2 - 4}$

30) $f(x) = \dfrac{4x}{x^2 + 4}$

Use the chart to find $h'(4)$

$f(4)$	$f'(4)$	$g(4)$	$g'(4)$
-8	3	3π	4

31) $h(x) = 5f(x) - \dfrac{2}{3}g(x)$

32) $h(x) = 3 + 8f(x)$

33) $h(x) = f(x)g(x)$

34) $h(x) = \dfrac{f(x)}{g(x)}$

35) $h(x) = \dfrac{g(x)}{f(x)}$

36) $h(x) = \dfrac{f(x) + 2}{-3g(x)}$

For each of the following, find $f''(x)$.

37) $f(x) = \dfrac{x^3 - 3x^2 - 4x - 1}{2x}$

38) $f(x) = \dfrac{x}{x - 4}$

39) $f(x) = \sqrt{x} - 4\sqrt[3]{x} + \dfrac{6}{5\sqrt[4]{x}}$

40) Find an equation of the line that is tangent to $f(x) = x^2 - 6x + 7$ and

a) parallel to the line $y = 2x + 4$

b) perpendicular to the line $y = 2x + 4$

Differentiation by the Chain Rule - Classwork

Suppose you were asked to take the derivatives of the following. Could you do so?

a) $f(x)=(2x+5)^2$ b) $f(x)=(2x+5)^3$ c. $f(x)=(2x+5)^6$ d) $f(x)=\sqrt{2x+5}$

a) causes no problem. b) is also not a problem but multiplying it out so you can take the derivative is a bit of a pain. You are capable of doing c) but clearly do not wish to. But d) can't be done with the knowledge you have.

We now introduce a method of taking derivatives of more complicated expressions. It is called the **chain rule**. If $y=f(u)$ is a differentiable function of u and $u=g(x)$ is a differentable function of x, then $y=f(g(x))$ is a differentable function of x and $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$ or equivalently, $\frac{d}{dx}\left[f(g(x))\right]=f'(g(x))\cdot g'(x)$.

Example 1) If $f(x)=(2x+5)^2$, find $f'(x)$ without and with the chain rule. Show they are equivalent.

a) without chain rule

b) with chain rule

Example 2) If $f(x)=(2x+5)^3$, find $f'(x)$ without and with the chain rule. Show they are equivalent.

a) without chain rule

b) with chain rule

Example 3) If $f(x)=(2x+5)^{10}$, find $f'(x)$

Example 4) If $f(x)=\sqrt{2x+5}$, find $f'(x)$

Example 5) Find y' if $y=\frac{1}{4x-3}$

Example 6) Find y' if $y=(3x^2-2x+1)^3$

Find the derivatives of the following:

7) $y = \left(7 - 4x^2\right)^{2/3}$ 8) $y = -5\sqrt{x^2 - 4x + 1}$ 9) $y = \dfrac{-2}{\sqrt[4]{6x-1}}$

More difficult problems: We now have 3 basic rules. Power rule, product rule, and quotient rule. Note that the chain rule is not a basic rule of differentiation. The chain rule is always in effect. Even when you find the derivative of $y = 7x$, your answer is 7 times the derivative of *x* which is 7(1) = 7.

Find the derivatives of the following:

10) $y = x^2(2x-3)^4$ 11) $y = x\sqrt{4 - x^2}$ 12) $y = \left(\dfrac{2x-1}{2x+1}\right)^5$

13) $y = \dfrac{x}{\sqrt{x^2-1}}$ 14) $y = \sqrt{\dfrac{x}{4x-1}}$ 15) $y = \left(x^2 - 4\right)\sqrt{x+2}$

Given that $f(2) = -3, f'(2) = 6, g(2) = 3, g'(2) = -2, f'(3) = 4$, find the derivatives of the following at $x = 2$.

16) $f(x) \cdot g(x)$ 17) $\dfrac{f(x)}{g(x)}$ 18) $\left[f(x)\right]^3$ 19) $f(g(x))$

Differentiation by the Chain Rule - Homework

Find the derivatives of the following:

1. $y = (3x-8)^4$

2. $y = (3x^2+2)^5$

3. $y = 4(x^2+x-1)^{10}$

4. $y = -5(4-9x)^{3/2}$

5. $y = \frac{1}{3x-2}$

6. $y = \frac{-1}{(x^2-5x-6)^2}$

7. $y = \left(\frac{2}{2-x}\right)^2$

8. $y = \frac{4x}{(x+1)^2}$

9. $y = \frac{-3}{(x^3-x^2+3)^3}$

10. $y = x^3(5x-1)^4$

11. $y = \sqrt{1-t}$

12. $y = \sqrt[3]{3x^3-4x+2}$

13. $y = \frac{2}{\sqrt{2x+3}}$

14. $y = \frac{-1}{\sqrt{x}+1}$

15. $y = \sqrt{\frac{3x}{2x-3}}$

16. $y=\sqrt{x}(1-2x)^2$

17. $y=\sqrt[3]{\dfrac{2t}{t^2-4}}$

18. $y=(x^2+2x-6)^2(1-x^3)^2$

For each of the following, find the equation of the tangent line at the indicated point. Verify by calculator.

19. $y=\sqrt{x^2+2x+8}$ at $(2,4)$

20. $y=\sqrt[5]{3x^3+4x}$ at $(2,2)$

21. $y=\sqrt{\dfrac{3x-1}{2x+1}}$ at $(-1,2)$

Given the following information, find the value of the derivative of the functions at $x=3$. Be careful, not all the information is needed to calculate these. Answers are next to the problem.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
3	1	8	-3	-5
6	3	-2	4	5
8	-1	3	π	4
1	2	-6	5	0

22. $f(x)+g(x)$ (Ans: -8)

23. $f(x)g(x)$ (Ans: -29)

24. $\dfrac{f(x)}{g(x)}$ (Ans: $\dfrac{-19}{64}$)

25. $\dfrac{g(x)}{f(x)}$ (Ans: 19)

26. $(f(x))^2$ (Ans: -6)

27. $\dfrac{1}{g(x)}$ (Ans: $\dfrac{5}{64}$)

28. $\sqrt{f(x)}$ (Ans: $\dfrac{-3}{2}$)

29. $\sqrt{f(x)+g(x)}$ (Ans: $\dfrac{-4}{3}$)

30 . $f^3(x)g(x)$ (Ans: -77)

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
3	1	8	-3	-5
6	3	-2	4	5
8	-1	3	π	4
1	2	-6	5	0

31. $\dfrac{1}{\sqrt[3]{g(x)}}$ (Ans: $\dfrac{5}{48}$)

32. $\dfrac{f(x)}{f(x)+g(x)}$ (Ans: $\dfrac{-19}{81}$)

33. $f(g(x))$ (Ans: -5π)

34. $g(f(x))$ (Ans: 0)

35. $f(f(x))$ (Ans: -15)

36. $g(g(x))$ (Ans: -20)

37. The table below gives some values of the derivative of some function f. Complete the table by finding (if possible) the derivatives of each of the following transformations of f.

a) $g(x) = f(x) - 2$

b) $h(x) = 2f(x)$

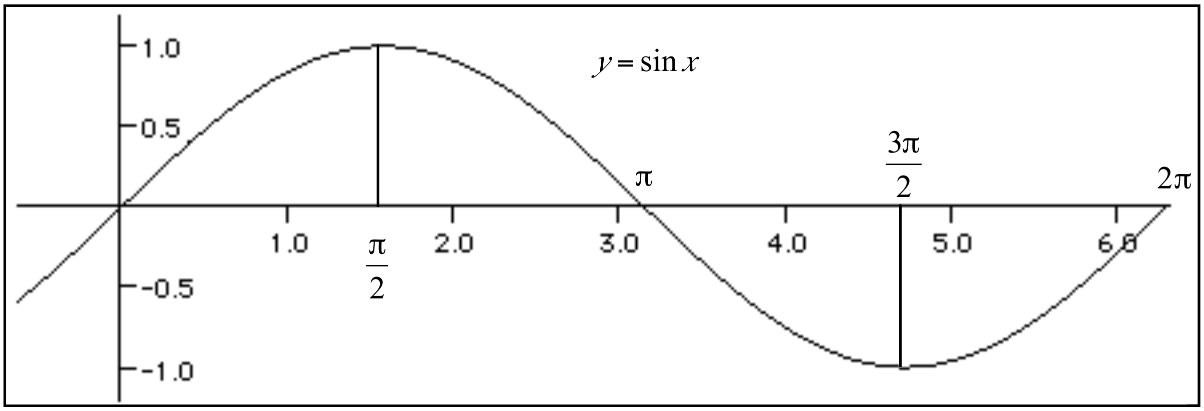
c) $r(x) = f(-3x)$

d) $s(x) = f(2x+1)$

x	**-2**	**-1**	**0**	**1**	**2**	**3**
$f'(x)$	4	$2/3$	$-1/3$	-1	-2	-4
$g'(x)$						
$h'(x)$						
$r'(x)$						
$s'(x)$						

Differentiation of Trigonometric Functions - Classwork

To find the derivative of $f(x)=\sin x$, we could try and go back to the basic definition of the derivative: $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$. Using that, we would get $f'(x)=\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$. From there, it is not clear how to proceed. So let us attempt to determine the derivative of $f(x)=\sin x$ by inspection. Below is a graph of $y=\sin x$. On the graph draw the tangent lines at $x=0$, $x=\frac{\pi}{2}$, $x=\pi$, $x=\frac{3\pi}{2}$, and $x=2\pi$.

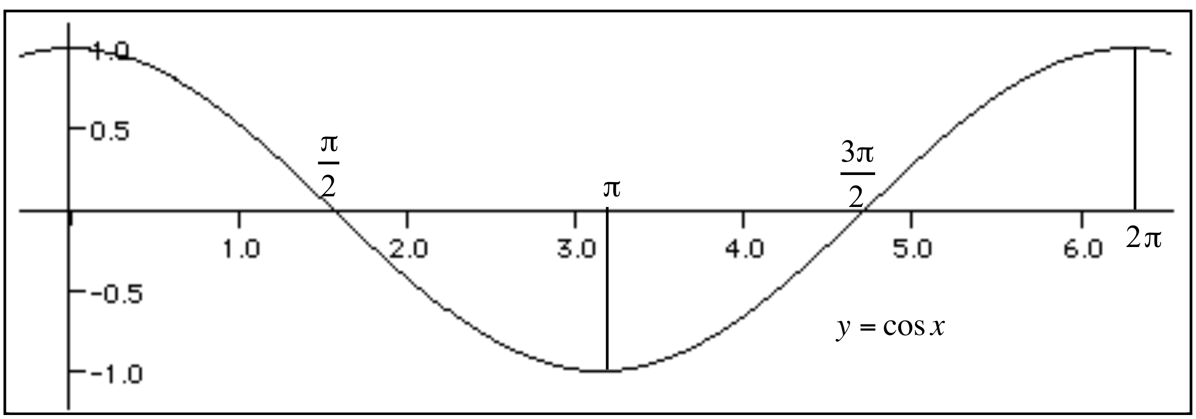


From your sketch of the tangent lines at these 5 points, deduce the slope of the tangent lines, i.e. the derivatives of sin *x* at these points, complete the chart below and then plot these points on the graph above.

x	0	$\pi/2$	π	$3\pi/2$	2π
$f'(x)$					

If you "connect the dots", what trig function does the derivative of $y=\sin x$ suggest? ___________________

Below is the graph of $y=\cos x$. Draw the tangent lines at $x=0$, $x=\frac{\pi}{2}$, $x=\pi$, $x=\frac{3\pi}{2}$, and $x=2\pi$, complete the chart, plot these points on this graph and "connect the dots."



x	0	$\pi/2$	π	$3\pi/2$	2π
$f'(x)$					

If you "connect the dots", what trig function does the derivative of $y=\cos x$ suggest? ___________________

We add to our basic knowledge of derivatives with these two definitions: You MUST memorize these.

$$\frac{d}{dx}[\sin x] = \cos x \qquad \text{and} \qquad \frac{d}{dx}[\cos x] = -\sin x$$

The derivatives of the other trig functions can be derived by using the ones above. Each proof makes use of the trig identity $\sin^2 x + \cos^2 x = 1$.

$y = \tan x$	$y = \csc x$	$y = \cot x$	$y = \sec x$
$y = \dfrac{\sin x}{\cos x}$	$y = \dfrac{1}{\sin x}$	$y = \dfrac{\cos x}{\sin x}$	$y = \dfrac{1}{\cos x}$
$y' = \dfrac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$	$y' = \dfrac{-\cos x}{\sin^2 x}$	$y' = \dfrac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x}$	$y' = \dfrac{-(-\sin x)}{\cos^2 x}$
$y' = \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x}$	$y' = \dfrac{-\cos x}{\sin x \cdot \sin x}$	$y' = \dfrac{-\sin^2 x - \cos^2 x}{\sin^2 x}$	$y' = \dfrac{\sin x}{\cos x \cdot \cos x}$
$y' = \dfrac{1}{\cos^2 x}$	$y' = \dfrac{-1}{\sin x} \cdot \dfrac{\cos x}{\sin x}$	$y' = \dfrac{-1}{\sin^2 x}$	$y' = \dfrac{1}{\cos x} \cdot \dfrac{\sin x}{\cos x}$
$y' = \sec^2 x$	$y' = -\csc x \cot x$	$y' = -\csc^2 x$	$y' = \sec x \tan x$

So the 6 trig function derivatives can be summarized:

$$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x$$

$$\frac{d}{dx}[\cos x] = -\sin x \qquad \frac{d}{dx}[\sec x] = \sec x \tan x$$

$$\frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\cot x] = -\csc^2 x$$

We now have 4 basic derivative forms: Power, product, quotient, and trig. In taking derivatives, we have to be careful to identify the type. Just because there is a power in the problem does not mean that the power rule is applied. Just because there is a trig function in the problem does not mean that the trig rules are to be applied.

Let us start by taking derivatives of the following. The first thing to do is to introduce parentheses for clarity. Let us also define the form of the problem (power, product, quotient, trig). The chain rule is used liberally in these problems.

a) $y = 2\sin x$ b) $y = \sin 2x$ c) $y = 2\sin 2x$ d) $y = \sin^2 x$

e) $y = \sin^2(2x)$ f) $y = \sin\sqrt{x}$ g) $y = \sqrt{\sin x}$ h) $y = \sqrt{\sin 2x}$

Take the derivatives of the following functions. Identify the form of the problem first.

1) $f(x) = 3\cos x$

2. $f(x) = \sin x \cos x$

3) $f(t) = t\cos t$

4) $y = \sin x + \cos x$

5) $y = 6\cos x^2$

6) $y = 6\cos^2 x$

7) $y = 6\cos^2 x^2$

8) $y = \dfrac{1+\sin x}{\cos x}$

9) $y = \sqrt{\cos x}$

10) $y = \tan\sqrt{x}$

11) $y = \tan\sqrt{5x}$

12) $y = \sec^3(2x+1)$

13) $y = 4\sec 3x \tan 3x$

14) $y = \sin x(x + \cos x)$

15) $y = \sqrt[3]{\cos 2x}$

Differentiation of Trigonometric Functions - Homework

Take the derivatives of the following functions. Identify the form of the problem and rewrite with parentheses.

1. $y = \sin 3x$

2. $y = x \sin x$

3. $y = \cos\left(\dfrac{\pi}{2} - x\right)$

4. $y = \dfrac{\sin x}{x}$

5. $y = \dfrac{x}{\sin x}$

6. $y = x^3 \sin^2 x$

7. $y = \cos 2x - \sin 3x$

8. $y = \cos^4 x^4$

9. $y = \sin^2 x + \cos^2 x$

10. $y = \sqrt{\sin x + 2}$

11. $y = \tan\sqrt{3x - 1}$

12. $y = \sec\left(x^2 - 2x + 3\right)$

13) $y = \cot^4\left(\dfrac{x}{2}\right)$

14) $y = \dfrac{\sin x}{1 + \cos^2 x}$

15) $y = \sin(\cos x)$

Find the equation of the tangent line to the following curves at the indicated point. Confirm by calculator.

16) $y = \sin x \cos x$ at $(0,0)$

17) $y = \dfrac{2x}{\cos x}$ at $(0,0)$

18) $y = \sin x(\sin x + \cos x)$ at $\left(\dfrac{\pi}{4}, 1\right)$

Implicit Differentiation - Classwork

Suppose you were asked to find the slope of the tangent line to the curve $x^2 + y^2 = 25$ at the point (4, 3).

First, we can solve for y
Next, take the derivative:
and simplify
Now plug in the value of $x = 4$.

Clearly, we have a problem. We have a ± in our derivative. Which one is it? ______________________________

The process we have gone through is actually called *explicit differentiation*. This process is accomplished by solving for *y*, then taking the derivative. But suppose we were asked to find the slope of the tangent line to the curve $x^2 + y^3 + y = 2$. Clearly, it is difficult or possibly impossible to solve for *y*. Another technique is needed. That technique is called *implicit differentiation*. In implicit differentiation we take the derivative without solving for *y*.

Let us do the original problem, finding the slope of the tangent line to the curve $x^2 + y^2 = 25$ at the point (4, 3) using implicit differentiation.

Original equation: $x^2 + y^2 = 25$
Take derivative (remember chain rule)
Now plug in the point (4, 3)

Notice that we did not actually solve for the derivative before we plugged in the point. But if we needed to find a general statement for $\frac{dy}{dx}$, it would be:

The key to doing implicit differentiation is to use your derivative rules on every term in the function but remembering to use the chain rule on that term. In differentiating the expression $x^2 + y^2 = 25$, we write: $2x\frac{dx}{dx} + 2y\frac{dy}{dx} = 0$. The $\frac{dx}{dx}$ term equals one and is not necessary but it is best to put it down and then eliminate it. Frequent careless mistakes are made by forgetting this term.

Example 1) Find $\frac{dy}{dx}$ for $x^2 - y^2 = 16$ at $(5,-3)$

Example 2) Find $\frac{dy}{dx}$ for $xy + y = 9$ at $(2,3)$

Example 3) Find $\frac{dy}{dx}$ for $x^2y + xy^2 = 2x$ at $(1,1)$

Example 4) Find $\frac{dy}{dx}$ for $y + \sqrt{xy} = 4$ at $(2,2)$

Example 5) Find $\frac{dy}{dx}$ for $(x+y)^2 + y = 2$ at $(0,1)$

Example 6) Find $\frac{dy}{dx}$ for $x^2 + 4y^2 = 4$ at $(2,0)$

Example 7) Find $\frac{dy}{dx}$ for $\sin(xy) = 1$

Example 8) Given $x^2 + y^2 = 100$, find $\frac{d^2y}{dx^2}$

Example 9) Find the equation of the tangent line to $\sqrt{x} + \sqrt{y} - 1 = y$ at $(9,4)$

Example 10) Find the points where the curve $25x^2 + 16y^2 + 200x - 160y + 400 = 0$ has horizontal and vertical tangent lines.

Implicit Differentiation - Homework

1. Find $\frac{dy}{dx}$ for $xy = 4$ at $(-4,-1)$

2. Find $\frac{dy}{dx}$ for $x^2 - y^3 = 0$ at $(1,1)$

3. Find $\frac{dy}{dx}$ for $\sqrt{x} + \sqrt{y} = 9$ at $(16,25)$

4. Find $\frac{dy}{dx}$ for $x^3 - xy + y^2 = 4$ at $(0,-2)$

5. Find $\frac{dy}{dx}$ for $x^2y - xy^2 = -6$ at $(2,-1)$

6. Find $\frac{dy}{dx}$ for $(x+y)^3 = x^3 + y^3$ at $(-1,1)$

7. Find $\frac{dy}{dx}$ for $\sqrt{xy} = x - 2y$ at $(4,1)$

8. Find $\frac{dy}{dx}$ for $x\cos y = 1$ at $\left(2,\frac{\pi}{3}\right)$

9. Find $\frac{dy}{dx}$ for $x^3y - y = x$

10. Find $\frac{dy}{dx}$ for $\sin x + 2\cos 2y = 1$

11. Find $\frac{dy}{dx}$ for $2\sin x \cos y = 1$

12. Find $\frac{dy}{dx}$ for $\tan(x + y) = y$

13. Given $1 - xy = x - y$, find $\frac{d^2y}{dx^2}$

14. Find the equations of the lines both tangent and normal to $x^3 + y^3 = 2xy$ at (1, 1)

15. Find the points at which the graph of $x^2 + 4y^2 - 4x + 16y + 4 = 0$ has a vertical and horizontal tangent line.

<u>Continuity and Differentiability - Classwork</u>

Back in our precalculus days, we dabbled in the concept of continuity. We reached a very informal definition of continuity: a curve is continuous if you can draw it without taking your pencil from the paper. This is a good "loose" definition but when one examines it closely, it is filled with holes. For instance, we know that the function $f(x) = x^2$ is a parabola and the parabola is continuous. But can we really draw it in its entirety without taking our pencil from the paper? Try it.

So we need a definition of continuity that is better than the one given above. We will use the following definition for continuity of a function.

A function is continuous at *c* if <u>all three</u> of the following holds:

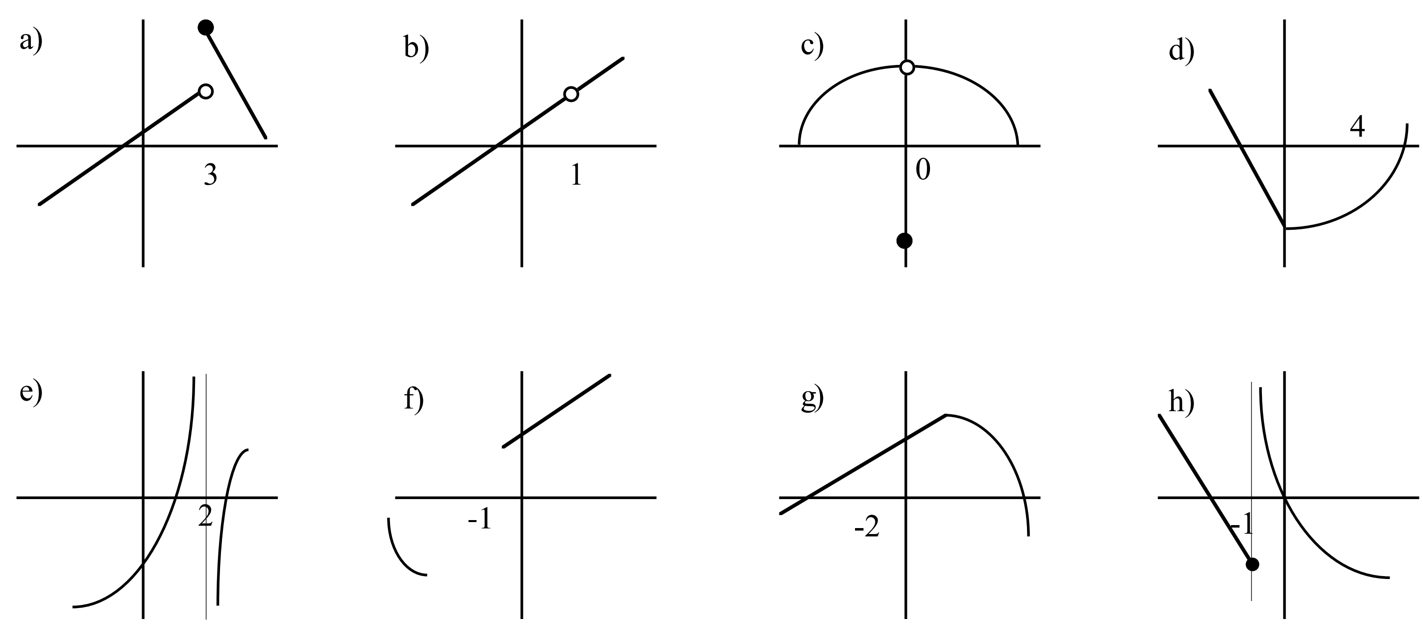
1. $\lim_{x \to c} f(x)$ exists	2. $f(c)$ exists	3. $\lim_{x \to c} f(x) = f(c)$

What we are saying is this: for the function to be continuous at some value of *c*:

1) as *x* gets closer and closer to *c* both on the left and right, we approach the same *y*-value.
2) the function is defined at $x = c$.
3) the *y*-value limit you got in step 1 is the same as the value you get for the function at *c* in step 2.

If a function is continuous at all values of *x* then we say it is a continuous function. A function can be continuous in certain parts of its domain and discontinuous in others.

Examples: In the following graphs determine if the function $f(x)$ is continuous at the marked value of *c*, and if not, determine which of the 3 rules of continuity the function fails.



Example a) above is an example of a "step" discontinuity. Notice how the function make a step at $x = 3$. Example b) is an example of a removable discontinuity (we usually call it a hole). We will see why it is called removable when we examine it in algebraic form.

When we examine functions in algebraic form, we can make the following conclusions:

a) all polynomials 1. $\lim_{x \to c} f(x)$ exists 2. $f(c)$ exists 3. $\lim_{x \to c} f(x) = f(c)$ are continuous at all values of x.

b) fractions in the form of $y = \dfrac{f(x)}{g(x)}$ are discontinuous wherever $g(x) = 0$.

c) radicals in the form of $y = \sqrt[\text{odd root}]{f(x)}$ are continuous everywhere.

d) radicals in the form of $y = \sqrt[\text{even root}]{f(x)}$ are discontinuous where $f(x) < 0$.

Examples: Find any points of discontinuity of the following functions.

a) $f(x) = -3x^2 - 5x + 1$ b) $f(x) = \dfrac{x-2}{x^2-4}$ c) $f(x) = \sqrt[3]{x^2 + 2x - 1}$ d) $f(x) = \sqrt{x^2 - x - 6}$

In dealing with continuity of a piecewise function, we need to examine the x-value where the rule changes.

Example 3) $f(x) = \begin{cases} x^2 - 3, x \ge 1 \\ 1 - x, x < 1 \end{cases}$

Example 4) $f(x) = \begin{cases} x^2 + 3x - 2, x \ge -2 \\ -x^2, x < -2 \end{cases}$

Example 5) $f(x) = \begin{cases} 3^{-x} - 1, x \ge -1 \\ \dfrac{1}{x+1}, x < -1 \end{cases}$

Example 6) $f(x) = \begin{cases} \dfrac{x-4}{x^2-16}, x \ne 4 \\ \dfrac{1}{3x-4}, x = 4 \end{cases}$

Find the value of the constant k that will make the function continuous. Verify by calculator.

Example 7) $f(x) = \begin{cases} 3x + 2, x \ge 1 \\ 2k - x, x < 1 \end{cases}$

Example 8) $f(x) = \begin{cases} kx^2, x \ge 2 \\ kx - 6, x < 2 \end{cases}$

Example 9) $f(x) = \begin{cases} k^2 - 12x, x \ge 1 \\ kx, x < 1 \end{cases}$

An important concept in calculus involves the concept of **differentiability**. There are several definitions that you need to know:

Definitions

Differentiability at a point: Function $f(x)$ is differentiable at $x = c$ if and only if $f'(c)$ exists. That is, $f'(c)$ is a real number.

Differentiability on an interval: Function $f(x)$ is differentiable on an interval (a,b) if and only if it is diffferentiable for every value of x on the interval (a,b).

Differentiability: Function $f(x)$ is differentiable if and only if it is differentiable at every value of x in its domain.

The concept of differentiability means, in laymans terms, "smooth." A differentiable curve will have no sharp points in it (cusp points) or places where the tangent line to the curve is vertical. Imagine a train traveling on a set of differentiable tracks and you will never get a derailment. Naturally, if a curve is to be differentiable, it must be defined at every point and its limit must exist everywhere. That implies the following:

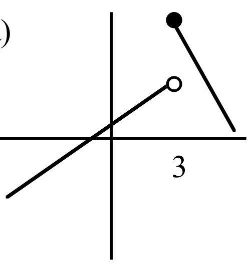
Differentiability implies Continuity, Continuity does not imply Differentiability.

If a function $f(x)$ is differentiable at $x = c$ then it must be continuous also at $x = c$. $D \Rightarrow C$

However, if a function is continuous at $x = c$, it need not be differentiable at $x = c$. Not! $C \Rightarrow D$

And, if a function is not continuous, then it can't be differentiable at $x = c$. not $C \Rightarrow$ not D

Example: determine whether the following functions are continuous, differentiable, neither, or both at the point.

a)


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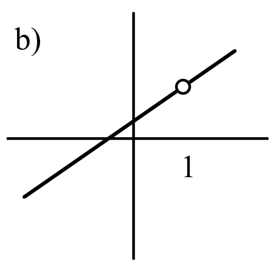
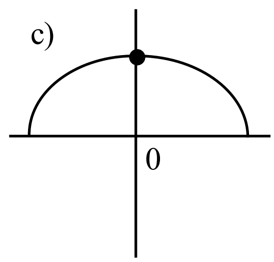


c)


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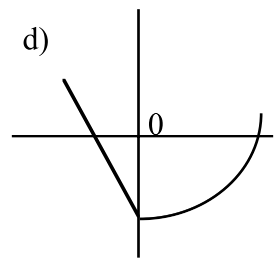
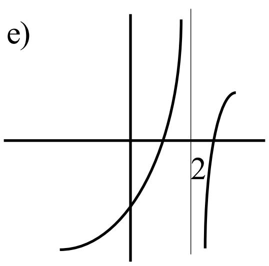


e)


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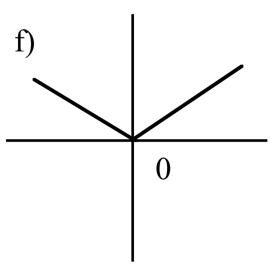
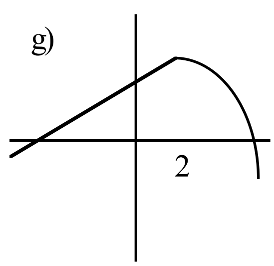


g)


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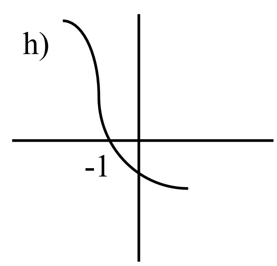


i) $f(x) = x^2 - 6x + 1$

j) $f(x) = \dfrac{x^2 - x - 12}{x + 3}$

k) $f(x) = \sin x$

l) $f(x) = \dfrac{\sin x}{x}$

Example 2) Determine if $f(x)$ is continuous and/or differentiable at the value of the function where the rule changes. Sketch the function.

a) $f(x)=\begin{cases} x^2-6x+10, x\geq 2 \\ 4-x, x<2 \end{cases}$

b) $f(x)=\begin{cases} x^2+x-3, x\geq -1 \\ -x-4, x<-1 \end{cases}$

c) $f(x)=\begin{cases} \sqrt{x+5}, x\geq 4 \\ 4-\sqrt[3]{x-4}, x<4 \end{cases}$

d) $f(x)=\begin{cases} \sin x, x\geq 0 \\ x-3x^2, x<0 \end{cases}$

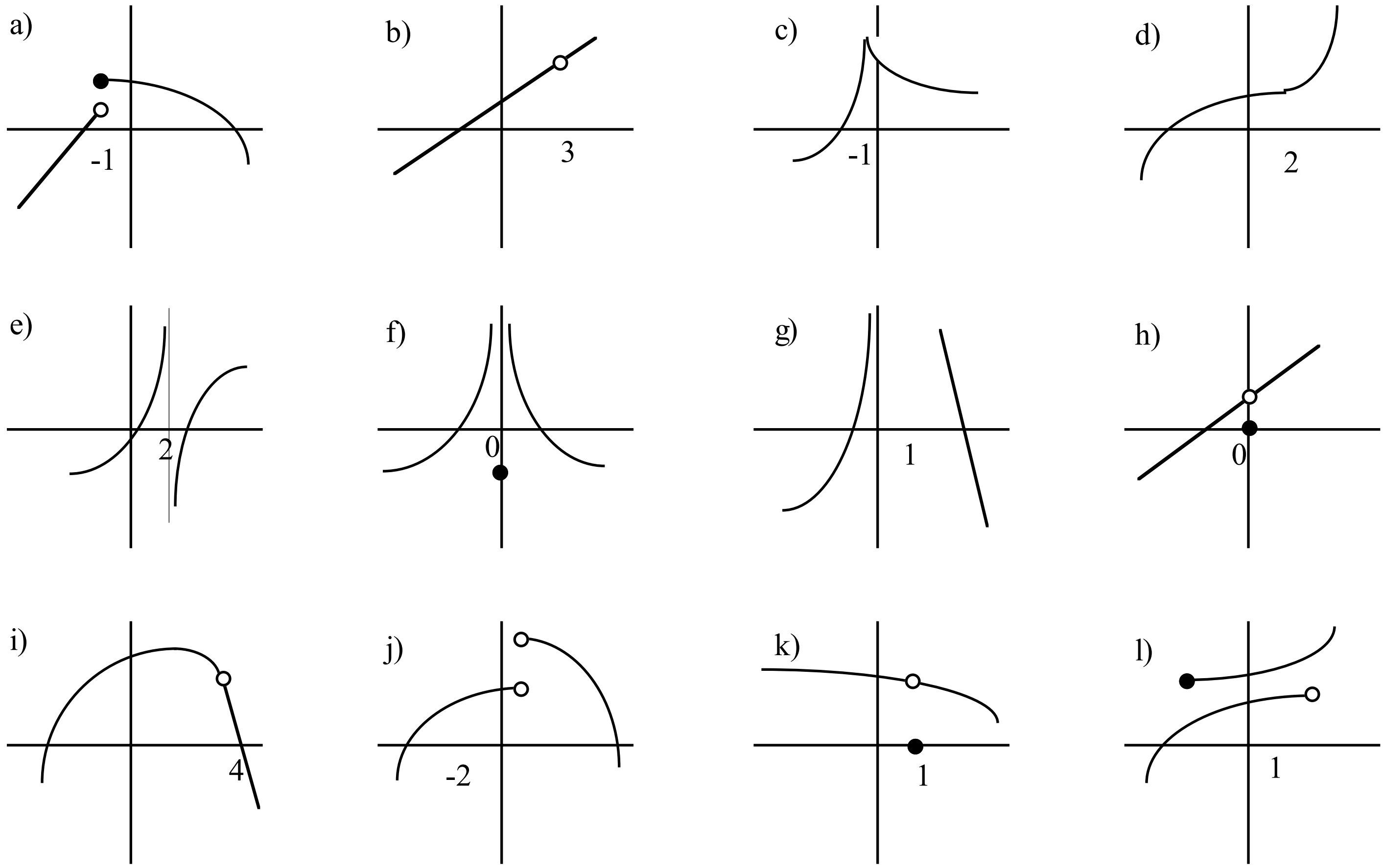
Example 3) Find the values of *a* and *b* that make the function $f(x)$ differentiable.

a) $f(x)=\begin{cases} ax^2+1, x\geq 1 \\ bx-3, x<1 \end{cases}$

b) $f(x)=\begin{cases} ax^3+1, x<2 \\ b(x-3)^2+10, x\geq 2 \end{cases}$

Continuity and Differentiability - Homework

1. In the following graphs determine if the function $f(x)$ is continuous at the marked value of *c*, and if not, determine for which of the 3 rules of continuity the function fails.



2. Find the value of *x* where the function is discontinuous.

a. $f(x) = x^3 + 3^x$ b. $f(x) = \dfrac{5}{x^2 - 81}$ c. $f(x) = \dfrac{x^2 + 2x - 24}{x^2 - 36}$ d. $f(x) = \tan x$

3. Find whether the function is continuous at the value where the rule for the function changes.

a. $f(x) = \begin{cases} 8 - x^2, x < 2 \\ 6 - x, x \geq 2 \end{cases}$ b. $f(x) = \begin{cases} 4 - x^2, x < 1 \\ 1 + x, x \geq 1 \end{cases}$ c. $f(x) = \begin{cases} 2^x, x < 3 \\ 10 - x, x \geq 3 \end{cases}$

d. $f(x)=\begin{cases} 2^{-x}, x<-1 \\ x+3, x\geq -1 \end{cases}$

e. $f(x)=\begin{cases} \dfrac{1}{x-2}, x<2 \\ 3, x=2 \\ x+1, x>2 \end{cases}$

f. $f(x)=\begin{cases} \dfrac{x^3-x}{x^2-x}, x\neq 0,\ x\neq 1 \\ 3, x=0 \\ 2, x=1 \end{cases}$

4. Find the value of the constant *a* that makes the function continuous.

a. $f(x)=\begin{cases} 0.4x+2, x>1 \\ 0.3x+a, x\leq 1 \end{cases}$

b. $f(x)=\begin{cases} x^2, x>2 \\ a-x, x\leq 2 \end{cases}$

c. $f(x)=\begin{cases} 9-x^2, x>2 \\ ax, x\leq 2 \end{cases}$

d. $f(x)=\begin{cases} ax+5, x<-1 \\ ax^2, x\geq -1 \end{cases}$

e. $f(x)=\begin{cases} 0.4x+a^2, x<-1 \\ ax+1.6, x\geq -1 \end{cases}$

f. $f(x)=\begin{cases} a^2-x^2, x<2 \\ 1.5ax, x\geq 2 \end{cases}$

5. Let *a* and *b* stand for constants and let $f(x)=\begin{cases} b-x, x<1 \\ a(x-2)^2, x\geq 1 \end{cases}$

a. Find an equation relating *a* and *b* if *f* is to be continuous at *x* = 1.

b. Find *b* if *a* = -1. Graph and show that the function is continuous

c. Find another value of *a, b* where *f* is continuous.

6. Graph the function $f(x)=x+4+\dfrac{10^{-25}}{x-2}$ and find what appears to be the limit of $f(x)$ as *x* approaches 2.

7. Sketch a function having the following attributes.

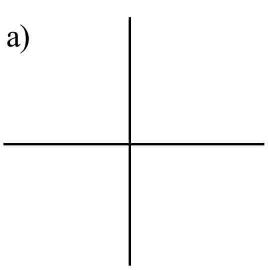
a) has a value of $f(2)$, a limit as *x* approaches 2, but is not continuous at *x* = 2.

b. has a step discontinuity at *x* = 3 where $f(3)=7$

c. $\lim\limits_{x\to 4} f(x)=-2$ but the function is not continuous at *x* = 4.

d. the value of $f(-2)=3$ but there is no limit of $f(x)$ as *x* approaches -2 and no vertical asymptote there.

a)



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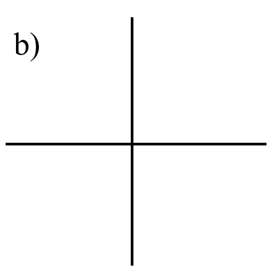

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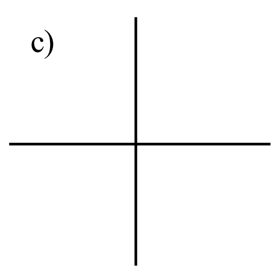

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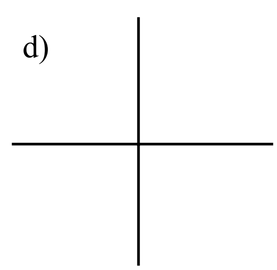
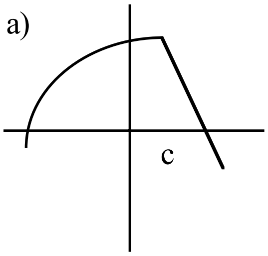


8. For the following, state whether the function is continuous, differentiable, both, or neither at $x = c$

a)



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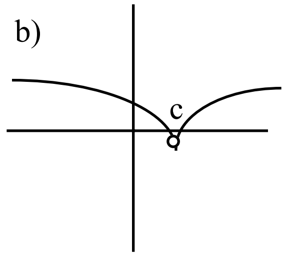

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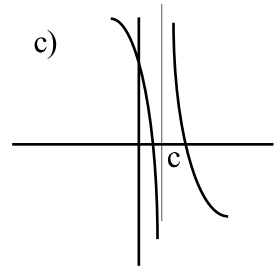

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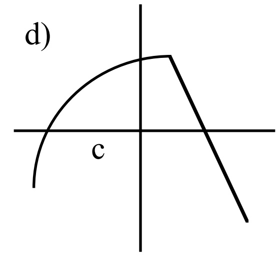

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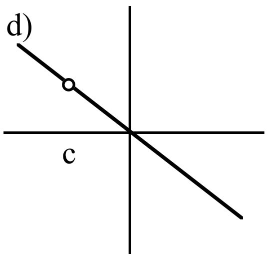

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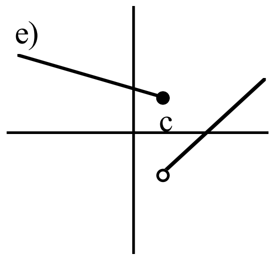

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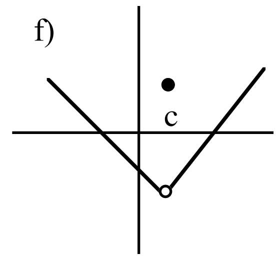

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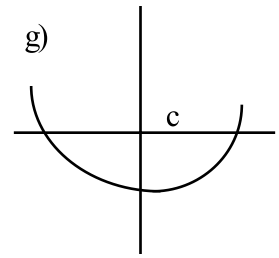


9. Sketch a function having the following attributes.

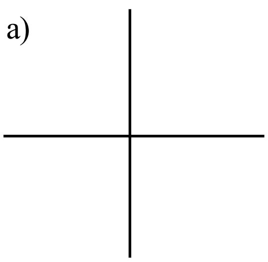
a. is differentiable and continuous at point (2, 4)

b. is continuous at (-3, 1) but not differentiable there

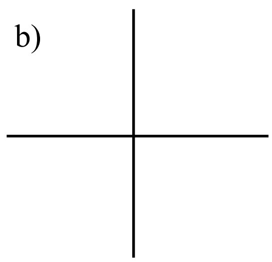
c. has a cusp at the point (-1, 3)

d. is differentiable at (2, -4) but not continuous there.

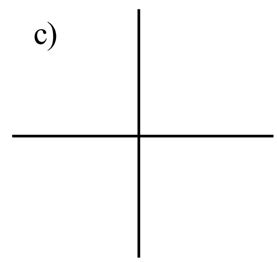
a)



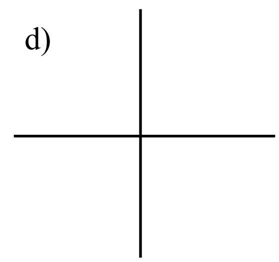
b)



c)



d)



10. For each function, $f(x)$, show work to determine whether the function is continuous or non-continuous, differentiable, or non-differentiable, and sketch the curve. Show work necessary to prove your statements.

a. $f(x) = \begin{cases} x^2, x \geq 0 \\ x, x < 0 \end{cases}$

b. $f(x) = \begin{cases} x^2 + 1, x \geq 0 \\ x^3 + 1, x < 0 \end{cases}$

c. $f(x) = \begin{cases} 4 - x^2, x < 1 \\ 2x + 2, x \geq 1 \end{cases}$

cont, diff, both, neither

cont, diff, both, neither

cont, diff, both, neither

d. $f(x)=\begin{cases} x^2+x-7, x\ge 2 \\ 5x-11, x<2 \end{cases}$

e. $f(x)=\begin{cases} x^4-2x^2, x>1 \\ -1, x\le 1 \end{cases}$

f. $f(x)=\begin{cases} \sqrt{x}-3, x>1 \\ \frac{1}{2}x-\frac{5}{2}, x\le 1 \end{cases}$

cont, diff, both, neither

cont, diff, both, neither

cont, diff, both, neither

g. $f(x)=\begin{cases} \sin(x), x>0 \\ x, x\le 0 \end{cases}$

h. $f(x)=\begin{cases} \cos(x), x\ge 0 \\ 1-x^2, x<0 \end{cases}$

i. $f(x)=\begin{cases} 3+(x+2)^{1/3}, x\ge -2 \\ 3-(x+2)^{2/3}, x<-2 \end{cases}$

cont, diff, both, neither

cont, diff, both, neither

cont, diff, both, neither

11. Find the values of *a* and *b* that make the function $f(x)$ differentiable.

a. $f(x)=\begin{cases} x^3, & x\ge 1 \\ a(x-2)^2+b, & x<1 \end{cases}$

b. $f(x)=\begin{cases} ax^2+10, & x\ge 2 \\ x^2-6x+b, & x<2 \end{cases}$

c. $f(x)=\begin{cases} a/x, & x\ge 1 \\ 12-bx^2, & x<1 \end{cases}$