# Optimization Problems - Homework

1. Find two numbers whose sum is 10 for which the sum of their squares is a minimum.

$$\boxed{\begin{array}{l} S = x^2 + (10 - x)^2 \\ S = 2x^2 - 20x + 100 \\ 4x - 20 = 0 \\ x = 5, y = 5 \end{array}}$$

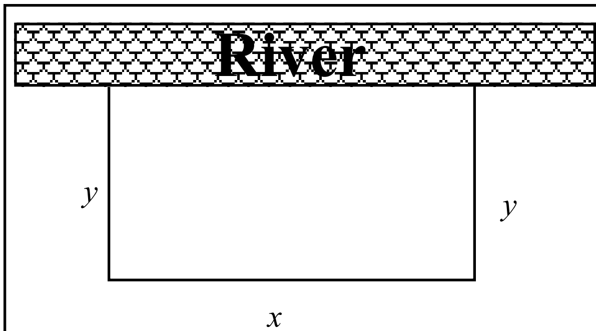
2. Find nonnegative numbers $x$ and $y$ whose sum is 75 and for which the value of $xy^2$ is as large as possible.

$$\boxed{\begin{array}{l} P = (75 - y)y^2 \\ P = 75y^2 - y^3 \\ 0 = 150y - 3y^2 \end{array}} \qquad \boxed{\begin{array}{l} 3y(50 - y) = 0 \\ y = 50, x = 25 \end{array}}$$

3. A ball is thrown straight up in the air from ground level. Its height after $t$ seconds is given by $s(t) = -16t^2 + 50t$. When does the ball reach it maximum height? What is its maximum height?

$$\boxed{\begin{array}{l} 0 = -32t + 50 \\ 32t = 50 \\ t = \dfrac{25}{16} \text{ sec, } s(t) = 39.063 \text{ ft} \end{array}}$$

4. A farmer has 2,000 feet of fencing to enclose a pasture area. The field will be in the shape of a rectangle and will be placed against a river where there is no fencing needed. What is the largest area field that can be created and what are its dimensions?




$$\begin{array}{l} P = x + 2y = 2000 \\ A = xy \\ A = (2000 - 2y)y \\ A = 2000y - 2y^2 \\ 0 = 2000 - 4y \\ 4y = 2000 \\ y = 500 \text{ ft}, x = 1000 \text{ ft}, \text{Area } = 500{,}000 \text{ ft}^2 \end{array}$$

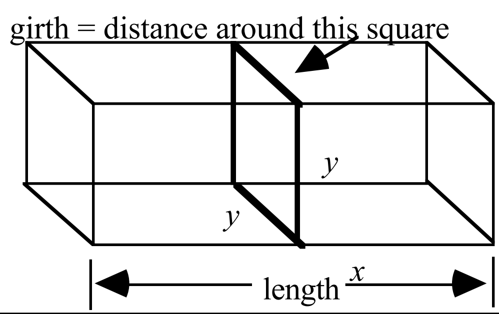
5. A fisheries biologist is stocking fish in a lake. She knows that when there are *n* fish per unit of water, the average weight of each fish will be $W(n) = 500 - 2n$, measured in grams. What is the value of *n* that will maximize the total fish weight after one season. *Hint: Total Weight = number of fish • average weight of a fish.*

$$W = n(500 - 2n)$$
$$W = 500n - 2n^2$$
$$0 = 500 - 4n$$
$$n = 125 \text{ fish}$$

6. The size of a population of bacteria introduced to a food grows according to the formula $P(t) = \dfrac{6000t}{60 + t^2}$ where *t* is measured in weeks. Determine when the bacteria will reach its maximum size. What is the maximum size of the population?

$$0 = \frac{(60 + t^2)(6000) - 6000t(2t)}{(60 + t^2)^2}$$
$$0 = 360000 - 6000t^2$$
$$t^2 = 60$$
$$t = \sqrt{60} \text{ - Week 8}$$
Size = 387 bacteria

7. The U.S. Postal Service will accept a box for domestic shipping only if the sum of the length and the girth (distance around) does not exceed 108 inches. Find the dimensions of the largest volume box with a square end that can be sent.

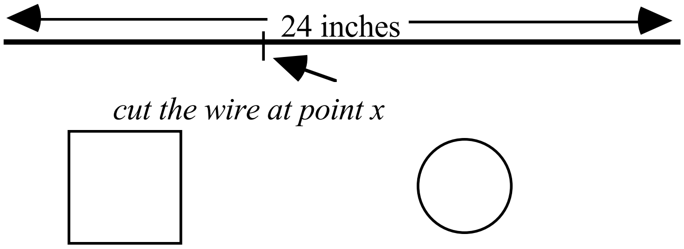



$$V = xy^2$$
$$V = (108 - 4y)y^2$$
$$V = 108y^2 - 4y^3$$
$$0 = 216y - 12y^2$$
$$0 = 12y(18 - y)$$
$$y = 18 \text{ in, } x = 36 \text{ in}$$

8. Blood pressure in a patient will drop by an amount $D(x)$ where $D(x) = 0.025x^2(30 - x)$ where *x* is the amount of drug injected in cm$^3$. Find the dosage that provides the greatest drop in blood pressure. What is the drop in blood pressure?

$$D(x) = 0.025x^2(30 - x)$$
$$D = .75x^2 - .025x^3$$
$$0 = 1.5x - .075x^2$$
$$0 = .075(20 - x)$$
$$x = 20 \text{ cm}^3, \text{ Drop } = 100 \text{ pts}$$

9. A wire 24 inches long is cut into two pieces. One piece is to be shaped into a square and the other piece into a circle. Where should the wire be cut to maximize the total area enclosed by the square and circle?




$$S = \frac{x^2}{16} + \pi\left(\frac{x^2 - 48x + 576}{4\pi^2}\right)$$

$$0 = \frac{x}{8} + \frac{2x - 48}{4\pi} \Rightarrow 0 = \pi x + 4x - 96$$

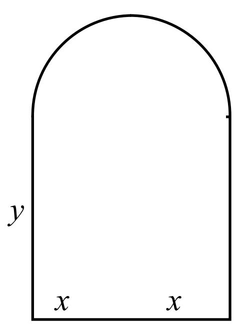
$$\pi x + 4x = 96 \Rightarrow x = \frac{96}{4 + \pi} = 13.442$$

This is a minimum - max occurs when it is all circle

Let $x$ be the point where the cut is made. Assume the square is on the left and the circle on the right. Complete the chart.

| $x$ | 4 | 8 | 12 | 20 | $x$ |
|---|---|---|---|---|---|
| Area square | 1 | 4 | 9 | 16 | $\left(\frac{x}{4}\right)^2$ |
| Area circle | $\frac{100}{4\pi}$ | $\frac{256}{4\pi}$ | $\frac{144}{4\pi}$ | $\frac{16}{4\pi}$ | $\frac{(24-x)^2}{4\pi}$ |
| Total area | 32.1 | 24.3 | 20.5 | 20.0 | |

10. A designer of custom windows wishes to build a Norman Window with a total outside perimeter of 60 feet. How should the window be designed to maximize the area of the window. A Norman Window contains a rectangle bordered above by a semicircle.




$$A = 2xy + \frac{1}{2}\pi x^2$$

$$A = 2x\left(\frac{60 - 2x - \pi x}{2}\right) + \frac{1}{2}\pi x^2$$

$$0 = 60 - 4x - \pi x$$

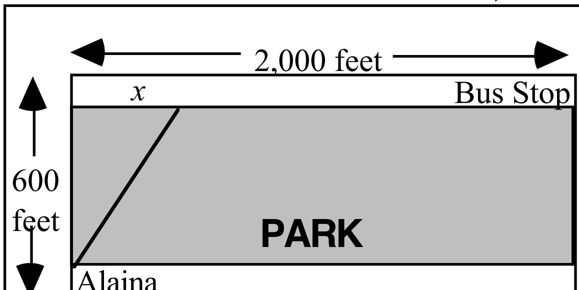
$$x = \frac{60}{\pi + 4} = 8.401$$

$$2x + 2y + \pi x = 60$$

$$y = \frac{60 - 2x - \pi x}{2}$$

Dimensions are 16.402 in. wide by 8.401 in. tall

11. Alaina wants to get to the bus stop as quickly as possible. The bus stop is across a grassy park, 2,000 feet east and 600 feet north of her starting position. Alaina can walk along the edge of the park on the sidewalk at a speed of 6 feet/sec. She can also travel through the grass in the park, but only at a rate of 4 ft/sec (dogs are walked here, so she must move with care). What path will get her to the bus stop the fastest.




$$T = \frac{\sqrt{x^2 + 360000}}{4} + \frac{2000 - x}{6}$$

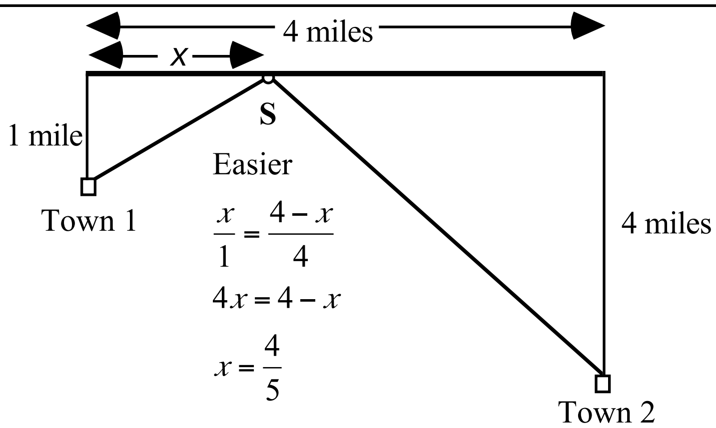
$$0 = \frac{x}{4\sqrt{x^2 + 360000}} - \frac{1}{6}$$

$$6x = 4\sqrt{x^2 + 360000}$$

$$9x^2 = 4x^2 + 1440000$$

$x = 536.656$ or $1463.344$ ft from bus stop

12. On the same side of a straight river are two towns, and the townspeople want to build a pumping station, **S**, that supplies water to them. The pumping station is to be at the river's edge with pipes extending straight to the two towns. The distances are shown in the figure below. Where should the pumping station be located to minimize the total length of pipe?

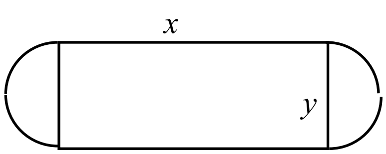



Easier

$$\frac{x}{1}=\frac{4-x}{4}$$
$$4x=4-x$$
$$x=\frac{4}{5}$$

$$D=\sqrt{x^2+1}+\sqrt{16-8x+x^2+16}$$
$$0=\frac{x}{\sqrt{x^2+1}}+\frac{x-4}{\sqrt{x^2-8x+32}}$$
$$-\frac{x}{\sqrt{x^2+1}}=\frac{x-4}{\sqrt{x^2-8x+32}}$$
$$\frac{x^2}{x^2+1}=\frac{x^2-8x+16}{x^2-8x+32}$$
$$x^4-8x^3+17x^2-8x+16=x^4-8x^3+32x^2$$
$$15x^2+8x-16=0$$
$$(5x-4)(3x+4)=0\Rightarrow x=\frac{4}{5}\text{ miles}$$

13. A physical fitness room consists of a rectangular region with a semicircle on each end. If the perimeter of the room is to be a 200-meter running track, find the dimensions that will make the area of the rectangular region as large as possible.




Total distance around track = 200 m

$$A=xy \qquad 200=4x \qquad 2x+\pi y=200$$
$$A=x\frac{(200-2x)}{\pi} \qquad x=50\text{ m} \qquad y=\frac{200-2x}{\pi}$$
$$0=\frac{200-2x^2}{\pi} \qquad y=\frac{100}{\pi}\text{ m}$$

14. Below is the graph of $y=1-x^2$. Find the point on this curve which is closest to the origin. (*Remember, you need a primary equation. What is it that you wish to minimize?*)

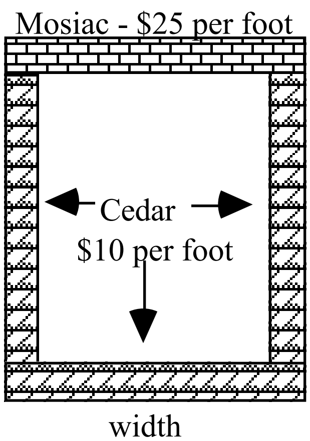

$$D=\sqrt{x^2+y^2}$$
$$D=\sqrt{x^2+\left(1-x^2\right)^2}$$
$$D=\sqrt{x^2+x^4-2x^2+1}$$
$$0=\frac{x\left(2x^2-1\right)}{\sqrt{x^4-x^2+1}}$$
$$2x^2=1$$
$$x=\sqrt{\frac{1}{2}},y=\frac{1}{2} \qquad \left(\sqrt{\frac{1}{2}},\frac{1}{2}\right)$$

# Economic Optimization Problems - Classwork

Example 1) A trucking company has determined that the cost per hour to operate a single truck is given by $C(s) = 0.0001s^2 - 0.01s + 112$ where $s$ is the speed that the truck travels. At what speed is the total cost per hour a minimum? What is the hourly cost to operate the truck?

$$\boxed{\begin{array}{l} 0 = .0002s - .01 \\ .0002s = .01 \end{array}} \qquad \boxed{s = 50 \text{ mph, Cost } = \$111.75}$$

Example 2) A nursery wants to add a 1,000-square-foot rectangular area to its greenhouse to sell seedlings. For aesthetic reasons, they have decided to border the area on three sides by cedar siding at a cost of \$10 per foot. The remaining side is to be a wall with a brick mosaic that costs \$25 per foot. What should the dimensions of the sides be so that the cost of the project will be minimized?




$$C = 10\left(x + \tfrac{2000}{x}\right) + 25x$$

$$C = 35x + \frac{20000}{x}$$

$$0 = 35 - \frac{20000}{x^2}$$

$$35x^2 = 20000$$

$$x = 23.905$$

Width = 23.905 ft, Length = 41.833 ft, Cost = \$1,673.32

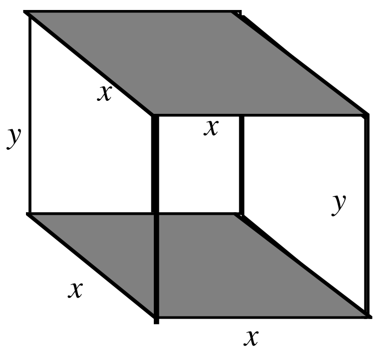
| Width | Length | Cedar Cost | Mosaic Cost | Total Cost |
|---|---|---|---|---|
| 10 | 100 | 2100 | 250 | 2350 |
| 100 | 10 | 1200 | 2500 | 3700 |
| 1000 | 1 | 10020 | 25000 | 35020 |
| $x$ | $\frac{1000}{x}$ | $10\left(x + \frac{2000}{x}\right)$ | $25x$ | $10\left(x + \frac{2000}{x}\right) + 25x$ |

Example 3) A real estate company owns 100 apartments in New York City. At \$1,000 per month, each apartment can be rented. However, for each \$50 increase, there will be two additional vacancies. How much should the real estate company charge for rent to maximize its revenues?

| **$50 increases** | **Rent** | **Apts. rented** | **Revenue** |
|---:|---:|---:|---:|
| 0 | 1000 | 100 | 100,000 |
| 1 | 1050 | 98 | 102,900 |
| 2 | 1100 | 96 | 105,600 |
| 3 | 1150 | 94 | 108,100 |
| 4 | 1200 | 92 | 110,400 |
| 50 | 3500 | 0 | 0 |
| $x$ | $1000 + 50x$ | $100 - 2x$ | $(1000 + 50x)(100 - 2x)$ |

$$\boxed{\begin{array}{l} R = (1000 + 50x)(100 - 2x) \\ R = 10000 + 3000x - 100x^2 \end{array}} \qquad \boxed{\begin{array}{l} 0 = 3000 - 200x \\ x = 15 \text{ Rent } = \$1,750 \end{array}}$$

Example 4) A closed box with a square base is to have a volume of 1,800 in$^3$. The material for the top and bottom of the box costs \$3 per square inch while the material for the sides cost \$1 per square inch. Find the dimensions of the box that will lead to the minimum total cost. What is the minimum total cost?




$C = 6x^2 + 4xy$ $\qquad x^2y = 1800$

$C = 6x^2 + 4x\left(\dfrac{1800}{x^2}\right)$ $\qquad y = \dfrac{1800}{x}$

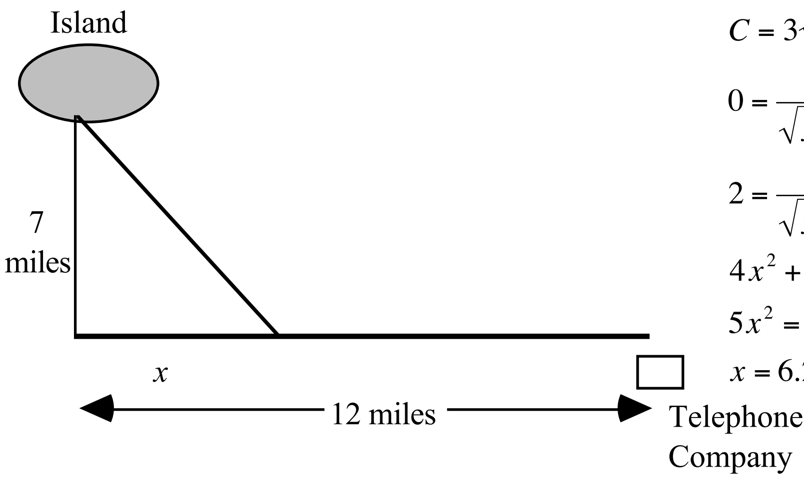
$0 = 12x - \dfrac{7200}{x^2}$

$12x^3 = 7200$

$x = \sqrt[3]{600}$

Base = 8.434 in, Height = 25.303 in, Cost = \$1,280.48

Example 5. A telephone wire is to be laid from the telephone company to an island 7 miles off shore at a cost of \$200,000 per mile along the shoreline and \$300,000 per mile under the sea. How should the wire be laid at the least expensive cost if the distance along the shoreline is 12 miles. What is that cost?




$C = 3\sqrt{x^2+49} + 2(12-x)$

$0 = \dfrac{3x}{\sqrt{x^2+49}} - 2$

$2 = \dfrac{3x}{\sqrt{x^2+49}}$

$4x^2 + 196 = 9x^2$

$5x^2 = 196$

$x = 6.261$ Miles, Cost = \$3,965,247.58

| x | water cost | land cost | total cost |
|---|---|---|---|
| 0 | 2.4 Million | 2.1 Million | 4.5 Million |
| 12 | 4.17 Million | 0 | 4.17 Million |
| $x$ | $3\sqrt{x^2+49}$ | $2(12-x)$ | $3\sqrt{x^2+49} + 2(12-x)$ |

Example 6) A small television company estimates that the cost (in dollars) of producing $x$ units of a certain product is given by $800 + .04x + .00002x^2$. Find the production level that minimizes the *average cost per unit.*

| Units | Cost | Average Cost |
|---|---|---|
| 100 | 904.20 | 9.04 |
| 1,000 | 960 | .96 |
| 5,000 | 1600 | .32 |
| 10,000 | 3300 | .33 |

$C = 800 + .04x + .00002x^2$

$AVG = \dfrac{800}{x} + .04 + .00002x$

$0 = \dfrac{-800}{x^2} + .00002$

$.00002x^2 = 800$

$\boxed{x \approx 6325 \text{ units}}$

# Economic Optimization Problems - Homework

1. The profit for Ace Advertising Co. is $P = 230 + 20s - \frac{1}{2}s^2$ where $s$ is the amount (in hundreds of dollars) spent on advertising. What amount of advertising gives the maximum profit?

$$\boxed{\begin{array}{l} P = 230 + 20s - \frac{1}{2}s^2 \\ 20 - s = 0 \\ s = 20 \quad \text{Amount is } \$2{,}000 \\ \text{Profit - } \$25{,}000 \end{array}}$$

2. North American Van Lines calculates charges for delivery according to the following rules.

Fuel cost $= \dfrac{v^2}{120}$ per hour $\qquad$ Driver cost = \$30 per hour

Find the speed $v$ that a truck should travel in order to minimze costs for a trip of 120 miles. *Hint: remember that rate • time = distance. Make a chart of possible speeds v and total costs.*

$$\boxed{\begin{array}{l} C = \frac{1}{120}\left(\frac{120}{t}\right)^2 t + 30t \\ C = \frac{120}{t} + 30t \end{array}}$$

$$\boxed{\begin{array}{l} 0 = \frac{-120}{t^2} + 30 \\ 30t^2 = 120 \\ t = 2 \text{ hrs} \Rightarrow \text{speed } = 60 \text{ mph, cost } = \$120 \end{array}}$$

3. Normally a pear tree will produce 30 bushels of pears per tree when 20 (or fewer) pear trees are planted per acre. However, for each additional pear tree planted above 20 trees per acre, the yield per tree will fall by one bushel per tree (why?). How many trees ahould be planted per acre to maximize the total yield? *Hint: Make a chart like the Apartment Housing sample problem.*

$$\boxed{\begin{array}{l} x = \text{number of pear trees above 20} \\ B = (30 - x)(20 + x) \\ B = 600 + 10x - x^2 \end{array}}$$

$$\boxed{\begin{array}{l} 0 = 10 - 2x \\ 2x = 10 \Rightarrow x = 5 \\ \text{25 trees} \end{array}}$$

4. Midas Muffler charges \$28 to replace a muffler. At this rate, the company replaces 75,000 mufflers per week nationally. For each additional dollar that the company charges, it tends to lose 1,000 customers a week. For each dollar the company subtracts from the \$28, the company gains 1,000 per week. How much should Midas charge to change a muffler in order to maximize their revenue? What would that revenue be? *Hint: Make a chart like the Apartment Housing sample problem.*

$$\boxed{\begin{array}{l} x = \text{number of dollars charged above } \$28 \\ R = (75000 - 1000x)(28 + x) \\ R = 2100000 + 47000x - 1000x^2 \\ 0 = 47000 - 2000x \end{array}}$$

$$\boxed{\begin{array}{l} 2000x = 47000 \\ x = 23.5 \\ \text{Charge } \$51.50, \text{ Revenue } = \$2{,}652{,}250 \end{array}}$$

5. A concert promoter knows that 5,000 people will attend an event with tickets set at \$50. For each dollar less in ticket price, an additional 1,000 tickets will be sold. What should the price of a ticket be in order to maximize the total receipts. *Hint: Make a chart like the Apartment Housing sample problem.*

| |
|---|
| $x$ = number of dollars less than \$50 |
| $R = (5000 + 1000x)(50 - x)$ |
| $R = 250000 + 45000x - 1000x^2$ |
| $0 = 45000 - 2000x$ |

| |
|---|
| $2000x = 45000$ |
| $x = 22.5$ |
| Charge \$28.50 Revenue = \$756,250 |

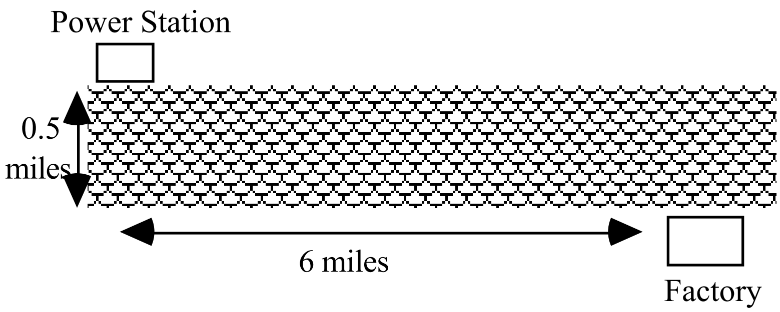
6. A travel agent is offering charter holidays in the Bahamas for college students. For groups of size up to 100, the fare is \$1,000 per student. For larger groups, the fare per person decreases by \$5 for each additional person in excess of 100. Find the size of the group that will maximize the travel agent's revenues. *Hint: Make a chart like the Apartment Housing sample problem.*

| |
|---|
| $x$ = number of additional people above 100 |
| $F = (100 + x)(1000 - 5x)$ |
| $F = 10000 + 500x - 5x^2$ |
| $0 = 500 - 10x$ |
| $10x = 500 \Rightarrow x = 50$ |
| Size = 150 at \$250 a student. Revenue is \$112,500. |

7. A real estate office handles 50 apartment units. When the rent is \$540 per month, all units are occupied. However, on the average, for each \$30 increase in rent, one unit becomes vacant. Each occupied unit requires an average of \$36 per month for service and repairs. What rent should be charged to realize the most profit?

| |
|---|
| $x$ = number of vacant units |
| $R = (50 - x)(540 + 30x) - 36(50 - x)$ |
| $R = 27000 + 960x - 30x^2 - 1800 + 36x$ |
| $0 = 960 - 60x + 36$ |
| $60x = 996 \Rightarrow x = 16.6 \approx 17$ |
| Rent = \$1050 a student. 33 apts rented. Revenue is \$33,462 |

8. A power station is on one side of a river that is .5 mile wide, and a factory is 6 miles downstream on the other side. It costs \$6,000 per mile to run power lines overland and \$8,000 per mile to run them underwater. Find the most economical path to lay transmission lines from the station to the factory.




$$C = 8\sqrt{x^2 + .25} + 6(6 - x)$$
$$0 = \frac{8x}{\sqrt{x^2 + .25}} - 6$$
$$3 = \frac{4x}{\sqrt{x^2 + .25}}$$
$$9x^2 + \frac{9}{4} = 16x^2$$
$$7x^2 = \frac{9}{4} \Rightarrow 28x^2 = 9$$
$x = .567$ miles from power station

9. A rectangular area is to be fenced in using two types of fencing. The front and back uses fencing costing \$5 a foot while the sides uses fencing costing \$4 a foot. If the area of the rectangle must contain 500 square feet, what should be the dimensions of the rectangle in order to keep the cost a minimum?

$$C = 5(2x) + 4(2y)$$
$$C = 10x + \frac{4000}{x}$$
$$0 = 10 - \frac{4000}{x^2}$$
$$10x^2 = 4000$$
$$x = 20$$

$$xy = 500$$
$$y = \frac{500}{x}$$

Front/back = 20 ft, Sides = 25 ft, Cost = \$400

10. The same rectangular area is to be built, but now the builder has only \$800 to spend. What is the *largest area* that can be fenced in using the same two types of fencing mentioned above.

$$A = xy$$
$$A = x\left(100 - \frac{5x}{4}\right)$$
$$A = 100x - \frac{5x^2}{4}$$
$$0 = 100 - \frac{5x}{2}$$
$$x = 40$$

$$10x + 8y = 800$$
$$y = 100 - \frac{5x}{4}$$

Front/back = 40 ft, Sides = 50 ft, Area = 2,000 $\text{ft}^2$

# Indefinite Integration - Classwork

Take a piece of notebook paper and cover up the paragraph under the chart below. Below, there are 5 terms. Write what you feel are the inverses of each of these terms.

| term | 5 | 1/3 | boy | dog | hot dog |
|---|---|---|---|---|---|
| **its inverse** | **nonsense** | **nonsense** | **nonsense** | **nonsense** | **nonsense** |

As ridiculous as the last problem is (how can you have an inverse of a hot dog?) so are they all ridiculous. The problem is that all of these terms aboves are nouns. Inverses refer not just to an opposite but to an opposite process. We take inverses of verbs, not nouns. Write the inverses of these processes.

| process | sit down | get dressed | take a book home | get wet | go to sleep |
|---|---|---|---|---|---|
| **its inverse** | **stand up** | **get undressed** | **take it to school** | **dry off** | **wake up** |

In mathematics, you have learned about operations on functions $f$. The inverse operation is denoted as $f^{-1}$ (not to be confused with $x^{-1}$, the reciprocal of *x* (remember *x* is a noun and $f$ is a process which is a verb.) When ever you perform an operation and immediately perform its inverse, you will end up exactly where you started with. We say that $f^{-1}\left[f(x)\right] = x$ and also $f\left[f^{-1}(x)\right] = x$. Below write some mathematical functions and their inverses.

| Function | Inverse |
|---|---|
| **addition** | **subtraction** |
| **multiplication** | **division** |
| **square** | **square root** |

| Function | Inverse |
|---|---|
| **log** | **$10^x$** |
| **trig** | **inverse trig** |
| **raising to n power** | **taking nth root** |

So, obviously since differentiation of functions is a process, we must have an inverse of that process. We call that process antidifferentiation. For instance, we know that the derivative of $y = x^3$ is $3x^2$. So it makes sense to say that an antiderivative of $3x^2$ is $x^3$.

However, it is important to say *an* antiderivative of $3x^2$ rather than *the* antiderivative of $3x^2$ for the simple reason that the derivative of $y = x^3$ is $3x^2$, but so is the derivative of $y = x^3 + 2$, $y = x^3 - 5$., and $y = x^3 + 6\pi$. There are an infinite number of functions whose derivative is $3x^2$. So when we go backwards to the antiderivative it is impossible to determine which function it came from. So to cover our bets, we say that the antiderivative of $3x^2$ is $x^3 + C$, where *C* represents a constant. We call *C* the constant of integration. It is important to attach the +*C* after every antiderivative. What you are doing is saying that the antiderivative is a family of functions rather than one specific function.

The process of taking antiderivatives is called *integration*, specifically *indefinite integration* because of the constant of integration *C*. So we do not have to write the word antiderivative again, we use a symbol to represent an antiderivative. That symbol is called an integral sign which is written like this: ∫. The way we write an integral is:

$\int f(x)\,dx = F(x) + C$. The $dx$ tells you what the important variable is when you are integrating just as you need to know what the important variable is when you differentiate $\left(\dfrac{dy}{dt} \text{ as opposed to } \dfrac{dy}{dx}\right)$.

So, since $\frac{d}{dx}(4x) = 4$, we will say that $\int 4\,dx = 4x + C$ and

and since $\frac{d}{dx}(x^2 + 3x - 1) = 2x + 3$, we will say that $\int (2x+3)\,dx = x^2 + 3x + C$

Just as we have derivative rules, we have a corresponding rule for integrals. Here are some basic integration rules.

| *Differentiation formula* | *Integration formula* |
|---|---|
| $\frac{d}{dx}[C] = 0$ | $\int 0\,dx = C$ |
| $\frac{d}{dx}[kx] = k$ | $\int k\,dx = kx + C$ |
| $\frac{d}{dx}[kf(x)] = kf'(x)$ a constant can be "factored out" | $\int kf(x)\,dx = k\int f(x)\,dx + C$ - factor out constant |
| $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$ the derivative of a sum is the sum of derivatives | $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx + C$ integral of a sum is the sum of the integrals |
| $\frac{d}{dx}[x^n] = nx^{n-1}$ - the power rule | $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ - the power rule reversed |

Examples - find the integral of each of the following:

1) $\int 7\,dx \boxed{= 7x + C}$

2) $\int x^5\,dx \boxed{= \frac{x^6}{6} + C}$

3) $\int x^{12}\,dx \boxed{= \frac{x^{13}}{13} + C}$

4) $\int (x^4 - x^2)\,dx = \boxed{\frac{x^5}{5} - \frac{x^3}{3} + C}$

5) $\int (t^3 + t + 1)\,dt = \boxed{\frac{t^4}{4} + \frac{t^2}{2} + t + C}$

6) $\int 3x^3\,dx = \boxed{\frac{3x^4}{4} + C}$

7) $\int (2x^2 - 7x - 8)\,dx$

$\boxed{\frac{2x^3}{3} - \frac{7x^2}{2} - 8x + C}$

8) $\int \left(\frac{3}{4}x^5 + \frac{5}{3}x^2 - \frac{x}{2}\right)dx$

$\boxed{\frac{x^6}{8} + \frac{5}{9}x^2 - \frac{x^2}{4} + C}$

9) $\int \left(\pi x + \frac{1}{\pi}\right)dx$

$\boxed{\frac{\pi x^2}{2} + \frac{x}{\pi} + C}$

10) $\int \frac{1}{x^2}\,dx = \boxed{-\frac{1}{x} + C}$

11) $\int \left(\frac{4}{x^3} - \frac{5}{x^4}\right)dx = \boxed{\frac{-2}{x^2} + \frac{5}{3x^3} + C}$

12) $\int \sqrt{x}\,dx = \boxed{\frac{2x^{3/2}}{3} + C}$

13) $\int \left(2\sqrt[3]{y} - 4\sqrt[4]{y}\right)dy$

$\boxed{\frac{3}{2}y^{4/3} - \frac{16}{5}y^{5/4} + C}$

14) $\int \left(\frac{1}{\sqrt{x}} - x^{2/3}\right)dx$

$\boxed{2x^{1/2} - \frac{3}{5}x^{5/3} + C}$

15) $\int \left(x^\pi + \sqrt{\pi}\right)dx$

$\boxed{\frac{x^{\pi+1}}{\pi+1} + \sqrt{\pi}x + C}$

In taking integrals, you may have to be clever. There are only a certain set of rules and if an integration problem doesn't fit one of the rules, you may have to change the expression so that it does. I call this a "bag of tricks."

Trick 1 - multiply out, then integrate

Trick 2 - Split into individual fractions, then integrate

16) $\int (2x-3)^2\,dx$

$$\boxed{\begin{array}{l}\int (4x^2 - 12x + 9)\,dx \\ \frac{4x^3}{3} - 6x^2 + 9x + C\end{array}}$$

17) $\int \frac{x^2 + 3x + 1}{x^4}\,dx$

$$\boxed{\begin{array}{l}\int (x^{-2} + 3x^{-3} + x^{-4})\,dx \\ \frac{-1}{x} - \frac{3}{2x^2} - \frac{1}{x^3} + C\end{array}}$$

18) $\int \frac{(2x-5)(3x+2)}{\sqrt{x}}\,dx$

$$\boxed{\begin{array}{l}\int (6x^{3/2} - 11x^{1/2} - 10x^{-1/2})\,dx \\ \frac{12}{5}x^{5/2} - \frac{22}{3}x^{3/2} - 20x^{1/2} + C\end{array}}$$

We took derivatives of trig functions earlier in the year. So, natually, we should be able to go backwards and take integrals involving trig functions.

| Differentiation formula | Integration formula |
|---|---|
| $\frac{d}{dx}[\sin x] = \cos x$ | $\int \cos x\, dx = \sin x + C$ |
| $\frac{d}{dx}[\cos x] = -\sin x$ | $\int \sin x\, dx = -\cos x + C$ |
| $\frac{d}{dx}[\tan x] = \sec^2 x$ | $\int \sec^2 x\, dx = \tan x + C$ |
| $\frac{d}{dx}[\csc x] = -\csc x \cot x$ | $\int \csc x \cot x\, dx = -\csc x + C$ |
| $\frac{d}{dx}[\sec x] = \sec x \tan x$ | $\int \sec x \tan x\, dx = \sec x + C$ |
| $\frac{d}{dx}[\cot x] = -\csc^2 x$ | $\int \csc^2\ dx = -\cot x + C$ |

Differentiation and integration using the sine and cosine functions occur in calculus all the time and students always get confused with signs. A good way to remember is to follow this chart.

| | | | |
|---|---|---|---|
| | sin | | |
| | ◊ | | |
| | cos | | |
| | ◊ | | When you differentiate, you follow the chart down. |
| Differentiation | – sin | Integration | |
| | ◊ | | When you integrate, you follow the chart up. |
| | – cos | | |
| | ◊ | | |
| | sin | | |

Examples - find the integral of each of the following:

20) $\int 4\sin x\, dx$

$\boxed{-4\cos x + C}$

21) $\int \frac{-2\cos x}{3}\, dx$

$\boxed{\frac{-2}{3}\sin x + C}$

22) $\int \frac{5}{\cos^2 x}\, dx$

$\boxed{\begin{array}{l}\int 5\sec^2 x\, dx\\ 5\tan x + C\end{array}}$

23) $\int (4\cos x - 9\sin x)\, dx$

$\boxed{4\sin x + 9\cos x + C}$

24) $\int \left(\frac{-\sin x}{\cos^2 x}\right) dx$

$\boxed{\begin{array}{l}\int -\sec x \tan x\, dx\\ -\sec x + C\end{array}}$

25) $\int \left(\theta^2 - 2\csc^2\theta\right) d\theta$

$\boxed{\begin{array}{l}\int \left(\theta^2 - 2\csc^2\theta\right) d\theta\\ \frac{\theta^3}{3} + 2\cot\theta + C\end{array}}$

If you were given the statement that $\frac{dy}{dx} = 4x$, we can cross multiply to get $dy = 4x\,dx$. We can now integrate each side of the equation to get $\int dy = \int 4x\,dx$. From there, we can solve for *y*.

$$\boxed{y + C_1 = 2x^2 + C_2 \Rightarrow y = 2x^2 + C}$$

The original statement $\frac{dy}{dx} = 4x$ is called a differential equation (DEQ). In a differential equation, you are given a statement about the derivative of $y$: $\frac{dy}{dx}$. Your goal is to solve for $y$. We have done so with the exception of the $+C$, the constant of integration. So we have a *general solution* of the DEQ. But suppose we were told that if $x = 0$, then $y = 5$. From there we can solve for $C$ and we will thus have the *specific solution* of the DEQ. Let's do so.

$$\boxed{y = 0 + C = 5 \text{ so } C = 5 \quad \text{and thus } y = x^2 + 5}$$

Example 18) Solve the differential equation.

$$f'(x) = 3x - 1, f(2) = 3$$

$$\boxed{\begin{array}{l} f(x) = \int (3x-1)\,dx \\ f(x) = \dfrac{3x^2}{2} - x + C \\ f(2) = 6 - 2 + C = 3 \Rightarrow C = -1 \\ f(x) = 3x^2 - x - 1 \end{array}}$$

Example 19) Solve the differential equation.

$$f'(x) = x^2 - 2x + 2, f(3) = -1$$

$$\boxed{\begin{array}{l} f(x) = \int (x^2 - 2x + 2)\,dx \\ f(x) = \dfrac{x^3}{3} - x^2 + 2x + C \\ f(3) = 9 - 9 + 6 + C = -1 \Rightarrow C = -7 \\ f(x) = \dfrac{x^3}{3} - x^2 + 2x - 7 \end{array}}$$

Example 20) Solve the differential equation.

$$f''(x) = 2, f'(4) = 1, f(-1) = 2$$

$$\boxed{\begin{array}{l} f'(x) = \int 2\,dx \\ f'(x) = 2x + C_1 \\ f'(4) = 8 + C_1 = 1 \\ C_1 = -7 \\ f'(x) = 2x - 7 \end{array}} \qquad \boxed{\begin{array}{l} f(x) = \int (2x-7)\,dx \\ f(x) = x^2 - 7x + C_2 \\ f(-1) = 1 + 7 + C_2 = 2 \\ C_2 = -6 \\ f(x) = x^2 - 7x - 6 \end{array}}$$

Example 21) Solve the differential equation.

$$f''(x) = 2x, f'(-5) = 30, f(2) = -1$$

$$\boxed{\begin{array}{l} f'(x) = \int 2x\,dx \\ f'(x) = x^2 + C_1 \\ f'(-5) = 25 + C_1 = 30 \\ C_1 = 5 \\ f'(x) = x^2 + 5 \end{array}} \qquad \boxed{\begin{array}{l} f(x) = \int (x^2 + 5)\,dx \\ f(x) = \dfrac{x^3}{3} + 5x + C_2 \\ f(2) = \dfrac{8}{3} + 10 + C_2 = -1 \\ C_2 = -\dfrac{41}{3} \\ f(x) = \dfrac{x^3}{3} + 5x - \dfrac{41}{3} \end{array}}$$

Example 22) Given that the graph of $f(x)$ passes through the point (1, 6) and that the slope of its tangent line at $(x, f(x))$ is $2x + 1$, find $f(6)$.

$$\boxed{\begin{array}{l} f'(x) = 2x + 1 \\ f(x) = \int (2x+1)\,dx \\ f(x) = x^2 + x + C \end{array}} \qquad \boxed{\begin{array}{l} f(1) = 1 + 1 + C = 6 \\ C = 4 \\ f(x) = x^2 + x + 4 \end{array}} \qquad \boxed{f(6) = x^2 + x + 4 = 46}$$

# Indefinite Integration - Homework

1. $\int -9\,dx$

$\boxed{-9x+C}$

2. $\int -5x\,dx$

$\boxed{-\frac{5x^2}{2}+C}$

3. $\int (6+2x)\,dx$

$\boxed{6x+x^2+C}$

4. $\int x^7\,dx$

$\boxed{\frac{x^8}{8}+C}$

5. $\int \left(x^4+x^3-x^2\right)dx$

$\boxed{\frac{x^5}{5}+\frac{x^4}{4}-\frac{x^3}{3}+C}$

6. $\int \left(3x^3-4x^2\right)dx$

$\boxed{\frac{3x^4}{4}-\frac{4x^3}{3}+C}$

7. $\int \left(\frac{2}{3}x^5-\frac{5}{2}x+\frac{1}{2}\right)dx$

$\boxed{\frac{x^6}{9}-\frac{5x^2}{4}+\frac{x}{2}+C}$

8. $\int \left(\frac{3}{x^4}\right)dx$

$\boxed{\frac{-1}{x^3}+C}$

9. $\int \left(2-\frac{1}{x^5}+\frac{7}{x^3}\right)dx$

$\boxed{2x+\frac{1}{4x^4}-\frac{7}{2x^2}+C}$

10. $\int 5\sqrt{x}\,dx$

$\boxed{\frac{10}{3}x^{\frac{3}{2}}+C}$

11. $\int 5\left(\sqrt[5]{x}\right)dx$

$\boxed{\frac{25}{6}x^{\frac{6}{5}}+C}$

12. $\int \left(x^{3/4}-\frac{1}{x^{3/4}}\right)dx$

$\boxed{\frac{4}{7}x^{\frac{7}{4}}-4x^{\frac{1}{4}}+C}$

13. $\int 3\sqrt[3]{x^2}\,dx$

$\boxed{\frac{9}{5}x^{\frac{5}{3}}+C}$

14. $\int (x-5)^2\,dx$

$$\int \left(x^2-10x+25\right)dx$$
$$\frac{x^3}{3}-5x^2+25x+C$$

15. $\int 4(3x-2)^3\,dx$

$$4\int \left(27x^3-54x^2+36x-8\right)dx$$
$$4\left(\frac{27x^4}{4}-18x^3+18x^3-8x+C\right)$$
$$27x^4-72x^3+72x^2-32x+C$$

16. $\int \frac{x^3-4x-1}{2x^3}\,dx$

$$\int \left(\frac{1}{2}-\frac{2}{x^2}-\frac{1}{2x^3}\right)dx$$
$$\frac{1}{2}x+\frac{2}{x}+\frac{1}{4x^2}+C$$

17. $\int t^2(3+t)^2\,dt$

$$\int \left(t^4+6t^3+9t^2\right)dt$$
$$\frac{t^5}{5}+\frac{3t^4}{2}+3t^3+C$$

18. $\int \frac{(3x-2)^2}{\sqrt{x}}\,dx$

$$\int \left(9x^{\frac{3}{2}}-12x^{\frac{1}{2}}+4x^{\frac{-1}{2}}\right)dx$$
$$\frac{18x^{\frac{5}{2}}}{5}-8x^{\frac{3}{2}}+8x^{\frac{1}{2}}+C$$

19. $\int \frac{3\cos x}{5}\,dx$

$\boxed{\frac{3}{5}\sin x + C}$

20. $\int (1 - 6\cos x)\,dx$

$\boxed{x - 6\sin x + C}$

21. $\int \left(\frac{1}{x^2} - \sin x\right) dx$

$\boxed{\frac{-1}{x} + \cos x + C}$

22. $\int \left(\sec^2 t + \cos t + 1\right) dt$

$\boxed{\tan t + \sin t + t + C}$

23. $\int \left(\sin^2 x + \cos^2 x\right) dx$

$$\int 1\,dx$$
$$x + C$$

24. $\int \frac{\sin x}{1 - \sin^2 x}\,dx$

$$\int \frac{\sin x}{\cos^2 x}\,dx$$
$$\int \tan x \sec x\,dx$$
$$\sec x + C$$

Solve the following differential equations.

25. $f''(x) = 2, f'(1) = 4, f(2) = -2$

$$f'(x) = \int 2\,dx$$
$$f'(x) = 2x + C_1$$
$$f'(1) = 2 + C_1 = 4$$
$$C_1 = 2$$
$$f'(x) = 2x + 2$$

$$f(x) = \int (2x + 2)\,dx$$
$$f(x) = x^2 + 2x + C_2$$
$$f(2) = 4 + 4 + C_2 = -2$$
$$C_2 = -10$$
$$f(x) = x^2 + 2x - 10$$

26. $f''(x) = 2x, f'(2) = -1, f(3) = 1$

$$f'(x) = \int 2x\,dx$$
$$f'(x) = x^2 + C_1$$
$$f'(2) = 4 + C_1 = -1$$
$$C_1 = -5$$
$$f'(x) = x^2 - 5$$

$$f(x) = \int \left(x^2 - 5\right) dx$$
$$f(x) = \frac{x^3}{3} - 5x + C_2$$
$$f(3) = 9 - 15 + C_2 = 1$$
$$C_2 = 7$$
$$f(x) = \frac{x^3}{3} - 5x + 7$$

27. $f''(x) = \frac{1}{x^{3/2}}, f'(4) = 2, f(0) = 1$

$$f'(x) = \int x^{\frac{-3}{2}}\,dx$$
$$f'(x) = -2x^{\frac{-1}{2}} + C_1$$
$$f'(2) = -1 + C_1 = 2$$
$$C_1 = 3$$
$$f'(x) = -2x^{\frac{-1}{2}} + 3$$

$$f(x) = \int \left(-2x^{\frac{-1}{2}} + 3\right) dx$$
$$f(x) = -4x^{\frac{1}{2}} + 3x + C_2$$
$$f(0) = 0 + C_2 = 1$$
$$C_2 = 1$$
$$f(x) = -4x^{\frac{1}{2}} + 3x + 1$$

28. $f''(x) = \cos x, f'(\pi) = 2, f(\pi) = -1$

$$f'(x) = \int \cos x\,dx$$
$$f'(x) = \sin x + C_1$$
$$f'(\pi) = 0 + C_1 = 2$$
$$C_1 = 2$$
$$f'(x) = \sin x + 2$$

$$f(x) = \int (\sin x + 2)\,dx$$
$$f(x) = -\cos x + 2x + C_2$$
$$f(\pi) = 1 + 2\pi + C_2 = -1$$
$$C_2 = -2 - 2\pi$$
$$f(x) = -\cos x + 2x - 2 - 2\pi$$

# u-Substitution - Classwork

When you take derivatives of more complex expressions, you frequently have to use the chain rule to differentiate. The integration equivalent of the chain rule is called *u-substitution*. *u-substitution* allows you integrate expressions which do not appear integratable.

1) $\int x(x^2-1)^5 dx$ Set up a $u = \boxed{x^2-1}$ Find $\frac{du}{dx} = \boxed{2x}$. Solve for $du = \boxed{2x\,dx}$

$\frac{1}{2}\int 2x(x^2-1)^5 dx$ You need to manufacture your $du$ in the original expression. So you will have to multiply by $\boxed{2}$ on the inside and thus multiply by $\boxed{\frac{1}{2}}$ on the outside.

$$\boxed{\frac{1}{2}\int u^5 du = \frac{1}{2}\left(\frac{u^6}{6}\right) = \frac{(x^2+1)^6}{12} + C}$$

Now change everything to *u*. Now integrate in terms of *u*.

Finally, change back to the variable *x* and add *C*.

2) $\int (3x-2)^4 dx$

$$\boxed{\frac{1}{3}\int u^4 du = \frac{u^5}{15} \quad \frac{(3x-2)^5}{15} + C}$$

$\boxed{u = 3x-2 \quad du = 3\,dx}$

3) $\int \sqrt{5x-2}\,dx$

$$\boxed{\frac{1}{5}\int u^{\frac{1}{2}} du = \frac{2u^{\frac{3}{2}}}{15} \quad \frac{2(5x-2)^{\frac{3}{2}}}{15} + C}$$

$\boxed{u = 5x-2 \quad du = 5\,dx}$

4) $\int 4(6x-1)^{2/3} dx$

$$\boxed{\frac{4}{6}\int u^{\frac{2}{3}} du = \frac{4}{6}\cdot\frac{3u^{\frac{5}{3}}}{5} \quad \frac{2(6x-1)^{\frac{5}{3}}}{5} + C}$$

$\boxed{u = 6x-1 \quad du = 6\,dx}$

5) $\int x\sqrt{x^2-2}\,dx$

$$\boxed{\frac{1}{2}\int u^{\frac{1}{2}} du = \frac{1}{2}\cdot\frac{2u^{\frac{3}{2}}}{3} \quad \frac{(x^2-2)^{\frac{3}{2}}}{3} + C}$$

$\boxed{u = x^2-2 \quad du = 2x\,dx}$

6) $\int x^2\sqrt{1-4x^3}\,dx$

$$\boxed{-\frac{1}{12}\int u^{\frac{1}{2}} du = \frac{-1}{12}\cdot\frac{2u^{\frac{3}{2}}}{3} \quad \frac{-(1-4x^3)^{\frac{3}{2}}}{18} + C}$$

$\boxed{u = 1-4x^3 \quad du = -12x^2\,dx}$

7) $\int \frac{x}{\sqrt[3]{2x^2-1}}\,dx$

$$\boxed{\frac{1}{4}\int u^{\frac{-1}{3}} du = \frac{1}{4}\cdot\frac{3u^{\frac{2}{3}}}{2} \quad \frac{3(2x^2-1)^{\frac{2}{3}}}{8} + C}$$

$\boxed{u = 2x^2-1 \quad du = 4x\,dx}$

8) $\int x^{1/2}(x^{3/2}+2)^9 dx$

$$\boxed{\frac{2}{3}\int u^9 du = \frac{2}{3}\cdot\frac{u^{10}}{10} \quad \frac{(x^{3/2}+2)^{10}}{15} + C}$$

$\boxed{u = x^{3/2}+2 \quad du = \frac{3}{2}x^{1/2}dx}$

9) $\int (x+2)\sqrt{x^2+4x-3}\,dx$

$$\boxed{\frac{1}{2}\int u^{\frac{1}{2}} du = \frac{1}{2}\cdot\frac{2u^{\frac{3}{2}}}{3} \quad \frac{(x^2+4x-3)^{\frac{3}{2}}}{3} + C}$$

$\boxed{u = x^2+4x-3 \quad du = 2(x+2)\,dx}$

10) $\int (x+2)\sqrt{x-4}\,dx$

$$\int (u+6)u^{\frac{1}{2}}\,du$$
$$\int \left(u^{\frac{3}{2}} + 6u^{\frac{1}{2}}\right) du$$
$$\frac{2}{5}u^{\frac{5}{2}} + 6\cdot\frac{2}{3}u^{\frac{3}{2}}$$
$$\frac{2}{5}(x-4)^{\frac{5}{2}} + 4(x-4)^{\frac{3}{2}} + C$$

$u = x-4 \qquad x = u+4$
$du = dx$

11) $\int \frac{x-5}{\sqrt{x-6}}\,dx$

$$\int (u+1)u^{\frac{-1}{2}}\,du$$
$$\int \left(u^{\frac{1}{2}} + u^{\frac{-1}{2}}\right) du$$
$$\frac{2}{3}u^{\frac{3}{2}} + 2u^{\frac{1}{2}}$$
$$\frac{2}{3}(x-6)^{\frac{3}{2}} + 2(x-6)^{\frac{1}{2}} + C$$

$u = x-6 \qquad x = u+6$
$du = dx$

12) $\int \frac{x^2}{\sqrt{x+1}}\,dx$

$$\int \left(u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + u^{-\frac{1}{2}}\right) du$$
$$\frac{2}{5}u^{\frac{5}{2}} - \frac{4}{3}u^{\frac{3}{2}} + 2u^{\frac{1}{2}}$$
$$\frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{4}{3}(x+1)^{\frac{3}{2}} + 2(x+1)^{\frac{1}{2}} + C$$

$u = x+1 \qquad x = u-1$
$du = dx$

13) $\int \cos 4x\,dx$

$$\frac{1}{4}\int \cos u\,du = \frac{1}{4}\cdot\sin u$$
$$\frac{\sin 4x}{4} + C$$

$u = 4x$
$du = 4\,dx$

14) $\int 3\sin(1-3x)\,dx$

$$\frac{3}{-3}\int \sin u\,du = -(-\cos u)$$
$$\cos(1-3x) + C$$

$u = 1-3x$
$du = -3\,dx$

15) $\int \sin^3 x\cos x\,dx$

$$\int u^3\,du = \frac{u^4}{4}$$
$$\frac{\sin^4 x}{4} + C$$

$u = \sin x$
$du = \cos x\,dx$

16) $\int \tan 10x\sec 10x\,dx$

$$\frac{1}{10}\int \tan u\sec u\,du = \sec u$$
$$\frac{\sec 10x}{10} + C$$

$u = 10x$
$du = 10\,dx$

17) $\int \tan^2 x\sec^2 x\,dx$

$$\int u^2\,du = \frac{u^3}{3}$$
$$\frac{\tan^3 x}{3} + C$$

$u = \tan x$
$du = \sec^2 x\,dx$

18) $\int \sin x\sqrt{\cos x}\,dx$

$$-\int u^{\frac{1}{2}}\,du = \frac{-2u^{\frac{3}{2}}}{3}$$
$$\frac{-2(\cos x)^{\frac{3}{2}}}{3} + C$$

$u = \cos x$
$du = -\sin x\,dx$

19) $\int \frac{\cos x}{\sqrt{1-\sin x}}\,dx$

$$-\int u^{\frac{-1}{2}}\,du = -2u^{\frac{1}{2}}$$
$$-2\sqrt{1-\sin x} + C$$

$u = 1-\sin x$
$du = -\cos x\,dx$

# u-Substitution - Homework

1. $\int \sqrt{x-2}\, dx$

$$\int u^{1/2} du \qquad u = x-2,\ du = dx$$
$$\frac{2}{3}u^{3/2} = \frac{2(x-2)^{\frac{3}{2}}}{3} + C$$

2. $\int (2x+3)^{11}\, dx$

$$\frac{1}{2}\int u^{11} du \qquad u = 2x-3,\ du = 2dx$$
$$\frac{1}{2}\left(\frac{u^{12}}{12}\right) = \frac{(2x+3)^{12}}{24} + C$$

3. $\int \sqrt{5x-1}\, dx$

$$\frac{1}{5}\int u^{1/2} du \qquad u = 5x-1,\ du = 5dx$$
$$\left(\frac{1}{5}\right)\left(\frac{2}{3}\right)u^{3/2} = \frac{2(5x-1)^{\frac{3}{2}}}{15} + C$$

4. $\int \sqrt[3]{6x+1}\, dx$

$$\frac{1}{6}\int u^{1/3} du \qquad u = 6x+1,\ du = 6dx$$
$$\frac{1}{6}\left(\frac{3}{4}\right)u^{4/3} = \frac{(6x+1)^{\frac{4}{3}}}{8} + C$$

5. $\int 5(3-4x)^{2/3}\, dx$

$$\frac{-1}{4}\int 5u^{2/3} du \qquad u = 3-4x,\ du = -4dx$$
$$\frac{-1}{4}(5)\left(\frac{3}{5}\right)u^{5/3} = \frac{-3(3-4x)^{\frac{5}{3}}}{4} + C$$

6. $\int \frac{dx}{(8x-1)^3}$

$$\frac{1}{8}\int u^{-3} du \qquad u = 8x-1,\ du = 8dx$$
$$\frac{1}{8}\left(\frac{u^{-2}}{-2}\right) = \frac{-1}{16(8x-1)^2} + C$$

7. $\int x(x^2+2)^6\, dx$

$$\frac{1}{2}\int u^{6} du \qquad u = x^2+2,\ du = 2x\, dx$$
$$\frac{1}{2}\left(\frac{u^7}{7}\right) = \frac{(x^2+2)^7}{14} + C$$

8. $\int 6x^2\sqrt{3x^3-1}\, dx$

$$\frac{1}{9}\int 6u^{1/2} du \qquad u = 3x^3-1,\ du = 9x^2\, dx$$
$$6\left(\frac{1}{9}\right)\left(\frac{2}{3}\right)u^{3/2} = \frac{4(3x^3-1)^{\frac{3}{2}}}{9} + C$$

9. $\int \left(1+\frac{1}{x}\right)^3\left(\frac{1}{x^2}\right) dx$

$$-\int u^{3} du \qquad u = 1+\frac{1}{x},\ du = \frac{-1}{x^2}\, dx$$
$$-\left(\frac{u^4}{4}\right) = \frac{-\left(1+\frac{1}{x}\right)^4}{4} + C$$

10. $\int x^{1/3}\left(x^{4/3}+9\right)^8\, dx$

$$\frac{3}{4}\int u^{8} du \qquad u = x^{4/3}+9,\ du = \frac{4}{3}x^{1/3}\, dx$$
$$\frac{3}{4}\left(\frac{u^9}{9}\right) = \frac{\left(x^{4/3}+9\right)^9}{12} + C$$

11. $\frac{2}{3}\int \sqrt{4-\frac{3}{5}x}\, dx$

$$\frac{2}{3}\left(\frac{-5}{3}\right)\int u^{1/2} du \qquad u = 4-\frac{3}{5}x,\ du = \frac{-3}{5}\, dx$$
$$\frac{2}{3}\left(\frac{-5}{3}\right)\left(\frac{2}{3}\right)u^{3/2} = \frac{-20\left(4-\frac{3}{5}x\right)^{\frac{3}{2}}}{27} + C$$

12. $\int (3x+15)\sqrt{x^2+10x+4}\, dx$

$$\frac{3}{2}\int (x+5)u^{1/2} du \qquad u = x^2+10x+4,\ du = 2(x+5)\, dx$$
$$\frac{3}{2}\left(\frac{2}{3}\right)u^{3/2} = \left(x^2+10x+4\right)^{\frac{3}{2}} + C$$

13. $\int (x+2)\sqrt{x-2}\,dx$

$$\int (u+2+2)u^{1/2}\,du \qquad u = x-2,\ du = dx \qquad x = u+2$$
$$\int \left(u^{3/2}+4u^{1/2}\right) du$$
$$\frac{2}{5}u^{5/2}+4\left(\frac{2}{3}\right)u^{3/2} = \frac{2(x-2)^{\frac{5}{2}}}{5}+\frac{8(x-2)^{\frac{3}{2}}}{3}+C$$

14. $\int \frac{x^2}{\sqrt{x-4}}\,dx$

$$\int (u+4)^2 u^{-1/2}\,du \qquad u = x-4,\ du = dx \qquad x = u+4$$
$$\int \left(u^2+8u+16\right)u^{-1/2}\,du$$
$$\int \left(u^{3/2}+8u^{1/2}+16u^{-1/2}\right) du$$
$$\frac{2}{5}u^{5/2}+8\left(\frac{2}{3}\right)u^{3/2}+16(2)u^{1/2}$$
$$\frac{2(x-4)^{5/2}}{5}+\frac{16(x-4)^{3/2}}{3}+32(x-4)^{1/2}+C$$

15. $\int \sin 5x\,dx$

$$\frac{1}{5}\int \sin u\,du \qquad u = 5x,\ du = 5\,dx$$
$$\frac{1}{5}(-\cos u) = \frac{-\cos 5x}{5}+C$$

16. $\int \cos\frac{x}{2}\,dx$

$$2\int \cos u\,du \qquad u = \frac{x}{2},\ du = \frac{1}{2}\,dx$$
$$2(\sin u) = 2\sin\frac{x}{2}+C$$

17. $\int \frac{1}{3}\sec^2 8x\,dx$

$$\frac{1}{3}\left(\frac{1}{8}\right)\int \sec^2 u\,du \qquad u = 8x,\ du = 8\,dx$$
$$\frac{1}{3}\left(\frac{1}{8}\right)(\tan u) = \frac{\tan 8x}{24}+C$$

18. $\int \sin 4x\cos 4x\,dx$

$$\frac{1}{4}\int \sin u\cos u\,du \qquad u = \sin 4x,\ du = 4\cos 4x\,dx$$
$$\frac{1}{4}\int u\,du = \frac{1}{4}\left(\frac{u^2}{2}\right) = \frac{\sin^2 4x}{8}+C \quad \text{or} \quad \frac{-\cos^2 4x}{8}+C$$

19. $\int \cos^3 x\sin x\,dx$

$$-\int u^3\,du \qquad u = \cos x,\ du = -\sin x\,dx$$
$$-\left(\frac{u^4}{4}\right) = \frac{-\cos^4 x}{4}+C$$

20. $\int \tan x\sec^2 x\,dx$

$$\int u\,du \qquad u = \tan x,\ du = \sec^2 x\,dx$$
$$\frac{u^2}{2} = \frac{\tan^2 x}{2}+C$$

21. $\int \sqrt{\cos 6x}\,\sin 6x\,dx$

$$\frac{-1}{6}\int u^{1/2}\,du \qquad u = \cos 6x,\ du = -6\sin x\,dx$$
$$\frac{-1}{6}\left(\frac{2}{3}\right)\left(u^{3/2}\right) = \frac{-\left(\cos(6x)\right)^{\frac{3}{2}}}{9}+C$$

22. $\int \frac{\sin x}{(4-\cos x)^3}\,dx$

$$\int u^{-3}\,du \qquad u = 4-\cos x,\ du = \sin x\,dx$$
$$\frac{u^{-2}}{-2} = \frac{-1}{2(4-\cos x)^2}+C$$

# Sigma Notation - Classwork

We will switch gears for a section and learn a completely different type of problem. It will be apparent within a few sections why we are seemingly learning something unrelated to integration.

Suppose you were asked to find the sum of the first 5 terms of the following sequence:

1 + 2 + 4 + … = ________ How did you arrive at the answer? ______________________________

The problem with writing such addition problems with the ellipsis (…) , is that the rule for each term is not apparent. We introduce notation called **sigma notation** for such problems using the Greek letter sigma $\sum$.

> The sum of $n$ terms $a_1 + a_2 + a_3 + \ldots + a_n$ is written as $\sum_{i=1}^{n} a_i$ where $i$ is the index of summation and $a_i$ is the $i$th term of the sum. Note that sigma notation does not help you to calculate the sum, only to write the sum.

Examples - Find the following sums.

1) $\boxed{\sum_{i=1}^{8} 3 = 24}$ 2. $\boxed{\sum_{i=1}^{6} i = 21}$ 3. $\boxed{\sum_{j=1}^{7} j^2 = 140}$ 4. $\boxed{\sum_{k=-2}^{3} k^3 = 27}$

> Since $\sum_{i=1}^{n} a_i$ represents a summation of numbers, we can apply basic properties of addition and multiplication.
>
> $\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i$ (meaning you can factor out $k$) $\sum_{i=1}^{n}\left[a_i \pm b_i\right] = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$ (write one sum as 2 sums)

Find the following sums (calculators allowed)

5) $\boxed{\sum_{i=1}^{7} 8i = 224}$ 6) $\boxed{\sum_{i=1}^{5}\left[5i-2\right] = 65}$ 7) $\boxed{\sum_{i=1}^{5}\left[(i+2)(i+3)\right] = 160}$ 8) $\boxed{\sum_{i=1}^{8} \frac{i}{3} = 12}$

9) $\boxed{\sum_{i=1}^{6} \frac{i-2}{i} = \frac{11}{10}}$ 10) $\boxed{\sum_{i=2}^{8} \sqrt{i^2-1} = 34.114}$ 11) $\boxed{\sum_{i=1}^{100} (-1)^i = 0}$ 12) $\boxed{\sum_{i=0}^{10} (-1)^i i^2 = 55}$

**Technology:** You can use your TI-84 to generate and add terms of these sequences. To create a sequence you will use the SEQ command found in 2nd LIST OPS. The format of this command is Seq(formula in *x*, *x*, starting *x*, ending *x*). For instance, problem 5 above $\sum_{i=1}^{7} 8i$ would be Seq(8X,X,1,7). This will generate the sequence. Now to add the terms, you use the sum command found in 2nd LIST MATH. Use Sum(Ans). You can do this in one fell swoop: Sum(Seq(8X,X,1,7)). You may only sum up to 999 terms.

As quick as you can, find the sum $\boxed{\sum_{i=1}^{15} i = 120}$.

Suppose you were asked to find the sum $\sum_{i=1}^{100} i$. Would you add all 100 terms? There must be an easier way.

$$\sum_{i=1}^{100} i = 1 + 2 + 3 + ... + 50 + 51 + ... + 98 + 99 + 100$$

Instead of adding your terms in sequential order, add the first plus the last, 2nd and next to last, etc. Each gives a sum of 101. Altogether you have 101 added 50 times or 50(101) = 5050.

There are formulas you can use to add many terms. While it is not necessary that you memorize the formulas, you will find them extremely useful for difficult summations.

$$\boxed{\sum_{i=1}^{n} c = cn \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}}$$

Examples - Find the following sums.

13) $\sum_{i=1}^{45} 3i = 3\frac{45(46)}{2}$

3105

14) $\sum_{i=1}^{30} i^2 = \frac{30(31)(61)}{6}$

9455

15) $\sum_{i=1}^{50} \left(2i^2 - 1\right)$

$2\frac{50(51)(101)}{6} - 50$

85,800

16) $\sum_{i=1}^{60} \left(i^2 - 4i + 2\right)$

$\frac{60(61)(121)}{6} - 4\frac{60(61)}{2} + 120$

66,610

17) $\sum_{i=1}^{38} (3i - 5)^2$

$9\frac{38(39)(77)}{6} - 30\frac{38(39)}{2} + 25(38)$

149,891

18) $\sum_{i=1}^{20} \left(i^3 - i^2 + i\right)$

$\frac{(20^2)(21^2)}{4} - \frac{20(21)(41)}{6} + \frac{20(11)}{2}$

41,440

# Sigma Notation - Homework

For each problem, determine the sum by **generating each term** and calculate using the calculator.

1) $\boxed{\sum_{k=1}^{6}(3k-2)=51}$

2) $\boxed{\sum_{j=1}^{60}2=120}$

3) $\boxed{\sum_{k=1}^{10}(k^2-1)=375}$

4) $\boxed{\sum_{k=1}^{10}(k-1)^2=285}$

5) $\boxed{\sum_{i=1}^{5}(i+1)(2i-3)=80}$

6) $\boxed{\sum_{i=1}^{7}\frac{i}{i+1}=\frac{1479}{280}=5.282}$

7) $\boxed{\sum_{i=1}^{6}\frac{4}{i^2+2}=2.839}$

8) $\boxed{\sum_{i=1}^{8}(-1)^i i^3=304}$

9) $\boxed{\sum_{i=3}^{9}\sqrt{2i}=23.889}$

10) $\boxed{\sum_{i=1}^{5}\left(i^3-(i+1)^2\right)=135}$

**Use your formulas and calculators to calculate the values of the following.**

11) $\boxed{\sum_{i=1}^{17}5i=765}$

12) $\boxed{\sum_{i=1}^{20}i^2=2870}$

13) $\boxed{\sum_{i=1}^{20}(i^2-1)=2850}$

14) $\boxed{\sum_{i=1}^{25}(i+4)^2=8525}$

15) $\boxed{\sum_{i=1}^{30}(i^3+i)=216{,}690}$

16) $\boxed{\sum_{i=1}^{30}(i^3-i^2)=206{,}770}$

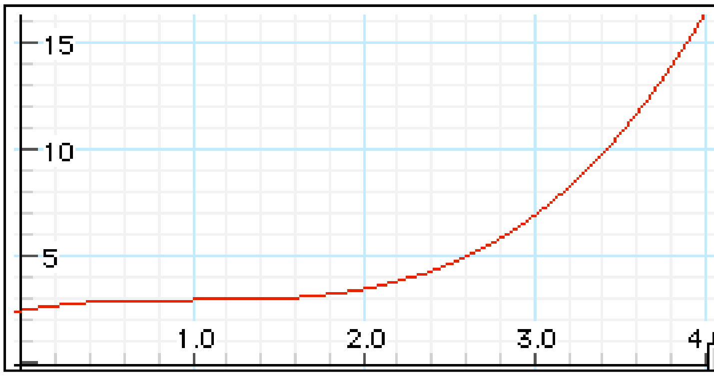
17) $\boxed{\sum_{i=1}^{10}(i^3+2i^2-5i+3)=3{,}550}$

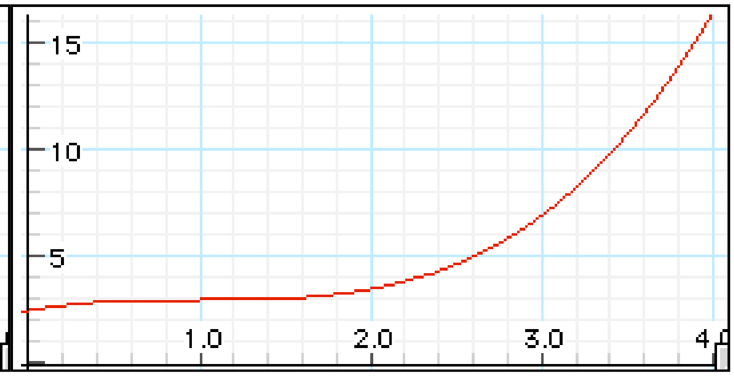
18) $\boxed{\sum_{i=5}^{30}(2i^3-i)=431{,}795}$

# Area Under Curve - Classwork

One of the basic problems of calculus is to find the slope of the tangent line (i.e. the derivative) at any point on the curve. The other basic problem is to find the area under the curve, that is the area between the curve and the $x$-axis between any two values of $x$.

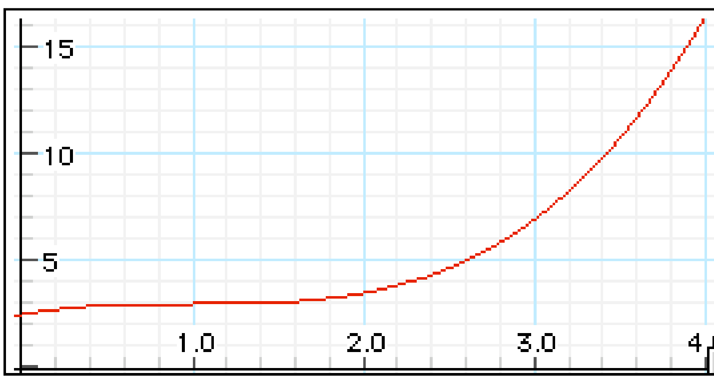
Below you are given a curve $y = f(x)$. Estimate what you think the area is. Then on the next three graphs, draw 2 rectangles, 4 rectangles, and 8 rectangles, total the areas of each and sum them for another estimate of the total area under the curve.

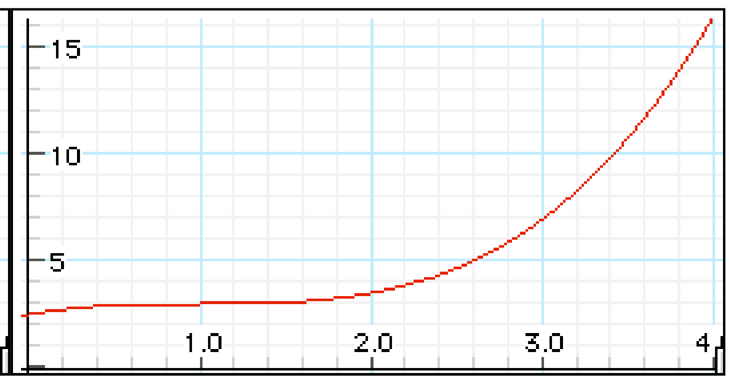





Estimate of area = **30**

2 Rectangles: **6** + **32** = **38**

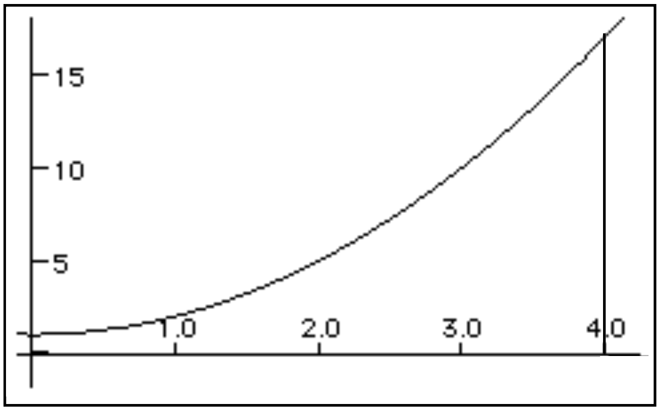


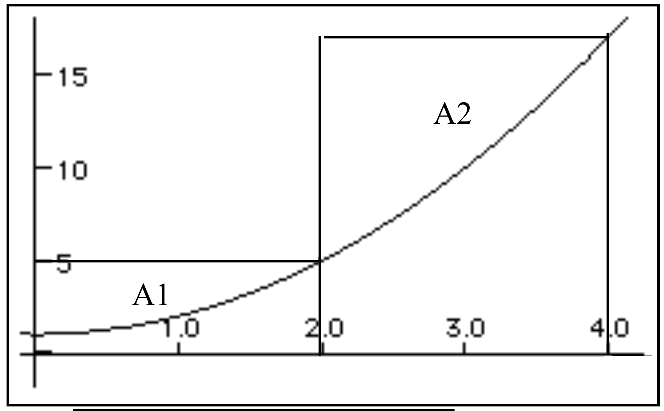



4 Rectangles: **3** + **3.5** + **7** + **16** = **29.5**

8 Rectangles: **1.5** + **1.5** + **1.5** + **2** + **2.3** + **3.5** + **5.0** + **8** = **25.3**

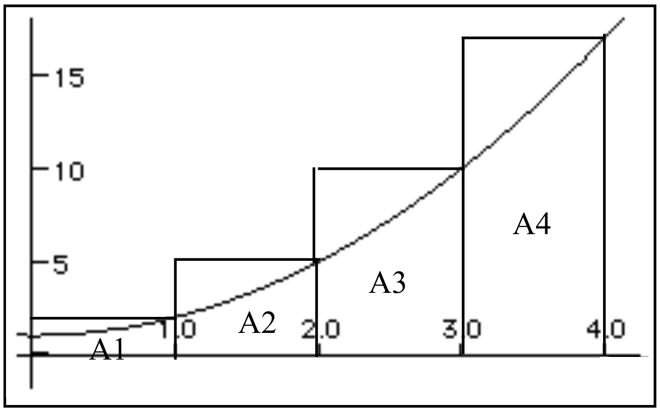
As you can see, as you use more rectangles, your estimate gets better but the amount of work you have to do increases. This is the idea we bring as we start our study of area. Now let's get more exact. Let's do the same thing with another curve, but this time you will be given the function: $f(x) = x^2 + 1$. Our problem is to estimate the area under the curve between $x = 0$ and $x = 4$ using "right" rectangles.

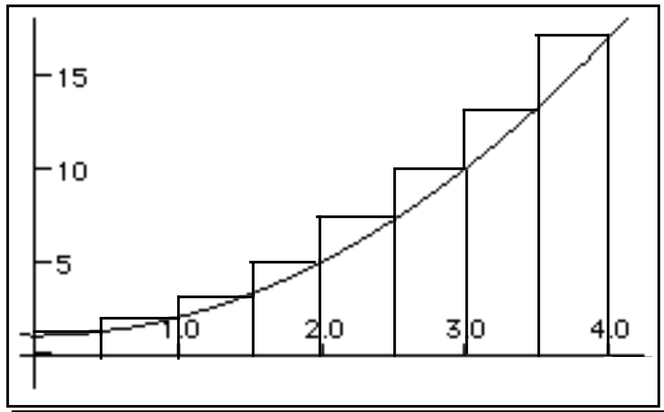



Estimate the area $\boxed{30}$

Area $\approx$ A1 + A2 $= b_1h_1 + b_2h_2$
Area $\approx b_1 f(2) + b_2 f(4) = 42$




Area $\approx$ A1 + A2 + A3 + A4
Area $\approx b_1h_1 + b_2h_2 + b_3h_3 + b_4h_4$
Area $\approx b_1 f(1) + b_2 f(2) + b_3 f(3) + b_4 f(4) = 34$

Area $\approx$ A1 + A2 + A3 + A4 + A5 + A6 + A7 + A8
Area $\approx b_1h_1 + b_2h_2 + b_3h_3 + b_4h_4 + b_5h_5 + b_6h_6 + b_7h_7 + b_8h_8$
Area $\approx b_1 f(.5) + b_2 f(1) + b_3 f(1.5) + b_4 f(2) +$
$b_5 f(2.5) + b_6 f(3) + b_7 f(3.5) + b_8 f(4) = 29.5$

As you go through this process, several things should be apparent:

- Drawing the function is not really necessary.
- The more rectangles you create, the more work you have to do. It is just a lot of arithmetic.
- In each case, the base is same allowing you to factor it out. For instance in the last case above,

$$b_1 f(.5) + b_2 f(1) + b_3 f(1.5) + b_4 f(2) + b_5 f(2.5) + b_6 f(3) + b_7 f(3.5) + b_8 f(4) =$$
$$b\left[f(.5) + f(1) + f(1.5) + f(2) + f(2.5) + f(3) + f(3.5) + f(4)\right]$$

- The more rectangles you create, the more accurate the area should be. So it should be apparent that

$$\text{True Area } = \lim_{n\to\infty} \sum_{i=1}^{n} A_i$$

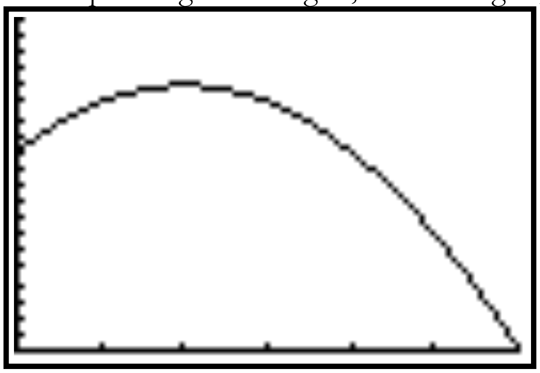
Examples) Find the area under the following functions using the indicated number of rectangles:

1) $f(x) = 3x + 1$ on [1,5]    2. $f(x) = x^2 + 3$ on [2,5]    3. $f(x) = x^2 - 3x - 2$ on [4,6]

a) 4 = $\boxed{46}$    b) 8 = $\boxed{43}$    a) 3 = $\boxed{59}$    b) 6 = $\boxed{53.375}$    a) 8 - $\boxed{18.438}$

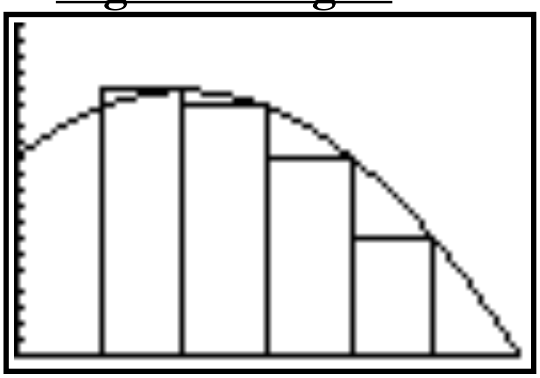
# Riemann Sums - Classwork

As a common example, this worksheet will use this problem. Find the area under the function $f(x)$ given in the picture below from $x = 1$ to $x = 5$. What we are looking for is the picture on the right. We will look at 4 techniques: right rectangles, left rectangles, midpoint rectangles and trapezoids.



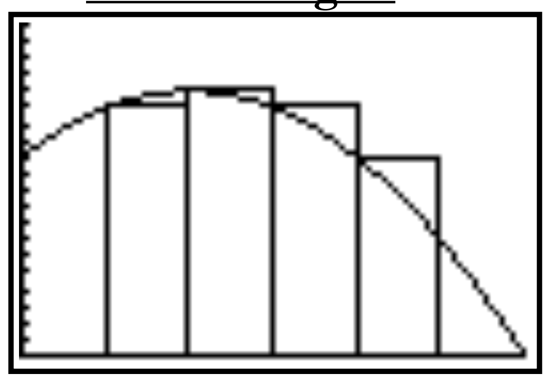
First some common statements. We will use 4 rectangles or trapezoids in this worksheet but you are expected to learn the technique for any number of rectangles or trapezoids. Obviously, if we wish 4 rectangles and the values of x run from 1 to 5, the base of each rectangle is 1. Here are the 4 pictures of what we are looking for.

## Right rectangles



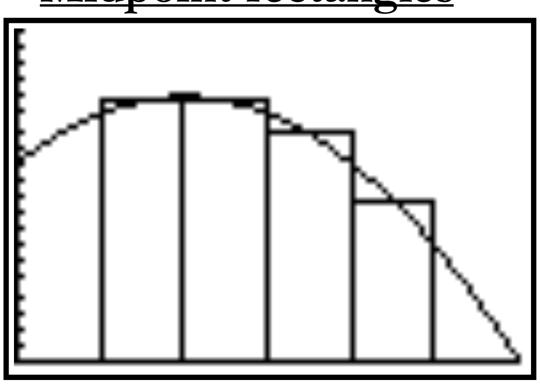
The height of the rectangle is on the right side.
This will underestimate the area (this case only).

## Left rectangles



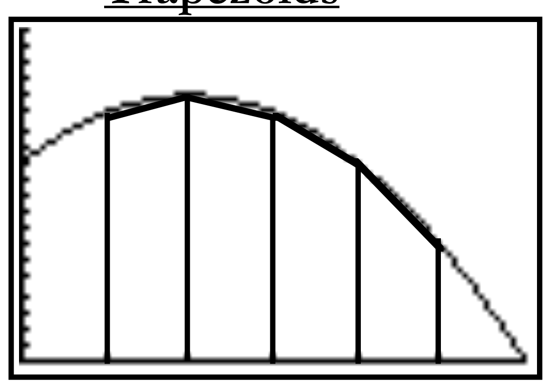
The height of the rectangle is on the left side.
This will overrestimate the area (this case only).

## Midpoint rectangles



The height of the rectangle is in the middle.
This ends up both over and underestimating the area.

## Trapezoids



The vertical lines represent the bases of the trapezoids
The result is a very good approximation to the area.

When we divide our picture into 4 rectangles, we have to find the base of each rectangle. In this case, since we were interested in doing this from $x = 1$ to $x = 5$, and there were 4 rectangles, we lucked out because the base of each rectangle is 1. That won't always happen. In general, let's call the base = $b$.

Now let us define $x_0, x_1, x_2, x_3 \ldots x_n$ as the places on the $x$-axis where we will build our heights, where $n$ represents the number of rectangles (or trapezoids). In this case,

$$x_0 = 1,\ x_1 = 2,\ x_2 = 3,\ x_3 = 4,\ x_4 = 5.$$

In the case of **left** rectangles, the area will be:

$$A \approx bh_0 + bh_1 + bh_2 + bh_3$$
$$A \approx b(h_0 + h_1 + h_2 + h_3)$$

In the case of **right** rectangles, the area will be:

$$A \approx bh_1 + bh_2 + bh_3 + bh_4$$
$$A \approx b(h_1 + h_2 + h_3 + h_4)$$

but since $h_i = f(x_i)$, we can say

$$A \approx b\big(f(x_0) + f(x_1) + f(x_2) + f(x_3)\big) \qquad A \approx b\big(f(x_1) + f(x_2) + f(x_3) + f(x_4)\big)$$

so, in the specific case above

$$A \approx 1\big(f(1) + f(2) + f(3) + f(4)\big) \qquad A \approx 1\big(f(2) + f(3) + f(4) + f(5)\big)$$

**so in general:**

$$A \approx b\sum_{i=0}^{n-1} f(x_i) \qquad A \approx b\sum_{i=1}^{n} f(x_i)$$

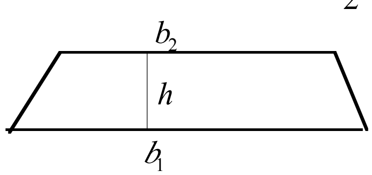
These are called <u>Riemann Sums</u>.

In the case of **midpoint rectangles**, you have to find the midpoint between your $x_0, x_1, x_2, x_3 \ldots x_n$
The midpoint between any two $x$ values is their sum divided by 2, so you will use:

$$A \approx b\left[f\left(\frac{(x_0 + x_1)}{2}\right) + f\left(\frac{(x_1 + x_2)}{2}\right) + f\left(\frac{(x_2 + x_3)}{2}\right) + f\left(\frac{(x_3 + x_4)}{2}\right)\right]$$

In our case, $A \approx 1\left[f\left(\frac{(1+2)}{2}\right) + f\left(\frac{(2+3)}{2}\right) + f\left(\frac{(3+4)}{2}\right) + f\left(\frac{(4+5)}{2}\right)\right]$ or $A \approx 1\big(f(1.5) + f(2.5) + f(3.5) + f(4.5)\big)$

For **trapezoids**, remember that area $A = \frac{1}{2} \cdot \text{height} \cdot (b_1 + b_2)$. That is when the trapezoid looks like this:



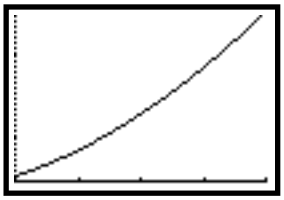

Since our traezoids are on their sides, we will say
$$A = \frac{1}{2} \cdot \text{base} \cdot (h_1 + h_2)$$

So, the total area $A \approx \frac{1}{2}b\left[\big(f(x_0) + f(x_1)\big) + \big(f(x_1) + f(x_2)\big) + \big(f(x_2) + f(x_3)\big) + \big(f(x_3) + f(x_4)\big)\right]$

or, in our case $A \approx \frac{1}{2}b\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right] = \frac{1}{2}b\left[f(1) + 2f(2) + 2f(3) + 2f(4) + f(5)\right]$

In general, the **trapezoidal rule**: $\boxed{A \approx \frac{1}{2}b\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + \ldots + 2f(x_{n-1}) + f(x_n)\right]}$

Let's try one: Let $f(x) = x^2 - 3$. We want to find the area under the curve using 8 rectangles/trapezoids from $x = 2$ to $x = 6$. First, let's draw it. Note that the curve is completely above the axis. If it dips below, the method changes slightly.



The drawing of the curve is helpful, but not necessary.

Since there are 8 rectangles, and we are finding the area between $x = 2$ and $x = 6$, the base is **.5**

Let's complete the chart:

| $i$ | $x_i$ | $f(x_i)$ |
|---|---|---|
| 0 | 2.0 | 1.00 |
| 1 | 2.5 | 3.25 |
| 2 | 3.0 | 6.00 |
| 3 | 3.5 | 9.25 |
| 4 | 4.0 | 13.00 |
| 5 | 4.5 | 17.25 |
| 6 | 5.0 | 22.00 |
| 7 | 5.5 | 27.25 |
| 8 | 6.0 | 33.00 |

So, the right rectangle formula gives $.5(3.25 + 6 + 9.25 + 13 + 17.25 + 22 + 27.25 + 33) = 65.5$

the left rectangle formula gives $.5(1 + 3.25 + 6 + 9.25 + 13 + 17.25 + 22 + 27.25) = 49.5$

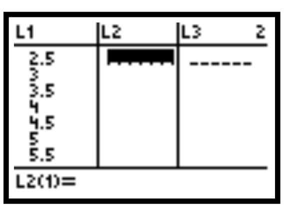
the trapezoid formula gives $\frac{.5}{2}(1 + 2 \cdot 3.25 + 2 \cdot 6 + 2 \cdot 9.25 + 2 \cdot 13 + 2 \cdot 17.25 + 2 \cdot 22 + 2 \cdot 27.25 + 33) = 57.5$

Note that the chart will not give you the midpoint formula. Let's do it here:

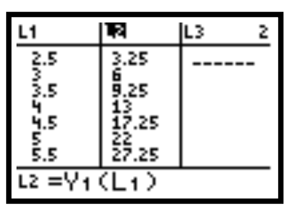
$$3.25 + 9.25 + 17.25 + 27.25 = 57$$

The calculator can generate this chart. Let's use right rectangles. Go to $\boxed{\text{STAT}}$ EDIT and clear out L1 and L2.

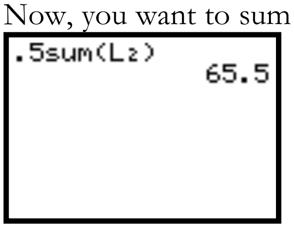
Place your $x_i$ in L1. It will look like this:




Now L2 contains $f(x_i)$. Since your function is in Y1, use




Now, you want to sum your L2 list and multiply it by your base which is .5. So go to your home screen and use:




You will find the SUM command in your $\boxed{\text{LIST}}$ MATH menu.

How do you adjust this for left rectangles? **replace the last data point 6 with the first data point 2**

# Interpretation of Area

1. A car comes to a stop 5 seconds after the driver slams on the brakes. While the brakes are on, the following velocities are recorded. Estimate the total distance the car took to stop.

| | | | | | | |
|---|---|---|---|---|---|---|
| Time since brakes applied (sec) | 0 | 1 | 2 | 3 | 4 | 5 |
| Velocity (ft/sec) | 88 | 60 | 40 | 25 | 10 | 0 |

$$\boxed{d \approx \frac{1}{2}(88 + 120 + 80 + 50 + 20 + 0) = 179 \text{ ft}}$$

2. You jump out of an airplane. Before your parachute opens, you fall faster and faster. Your acceleration decreases as you fall because of air resistance. The table below gives your acceleration $a$ (in m/sec$^2$) after $t$ seconds. Estimate the velocity after 5 seconds.

| | | | | | | |
|---|---|---|---|---|---|---|
| $t$ | 0 | 1 | 2 | 3 | 4 | 5 |
| $a$ | 9.81 | 8.03 | 6.53 | 5.38 | 4.41 | 3.61 |

$$\boxed{v \approx \frac{1}{2}(9.81 + 16.06 + 13.06 + 10.78 + 8.82 + 3.61) = 31.06 \text{ ft/sec}}$$

3. Cedarbrook golf course is constructing a new green. To estimate the area $A$ of the green, the caretaker draws parallel lines 10 feet apart and then measures the width of the green along that line. Determine how many square feet of grass sod that must be purchased to cover the green if

a) The caretaker is lazy and uses midpoint rectangles to calculate the area.
b) The caretaker uses left rectangles to calculate the area.
c) The caretaker uses right rectangles to calculate the area.
d) The caretaker uses trapezoids to calculate the area.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Width in feet | 0 | 28 | 50 | 62 | 60 | 55 | 51 | 30 | 3 |

a) midpoints: $A \approx A \approx 20(28 + 62 + 55 + 30) = 3500 \text{ ft}^2$
Note that you cannot assume that halfway between the first two given withs, the width will be 14. You can only use given data.

b) left: $\boxed{A \approx 10(28 + 50 + 62 + 60 + 55 + 51 + 30) = 3{,}360 \text{ ft}^2}$

c) right $\boxed{A \approx 10(28 + 50 + 62 + 60 + 55 + 51 + 30 + 3) = 3{,}390 \text{ ft}^2}$

d) trapezoids: $A \approx \frac{10}{2}(56 + 100 + 124 + 120 + 110 + 102 + 60 + 3) = 3{,}375 \text{ ft}^2$
Note that the trapezoid answer is the average of the left and right Riemann sums. This will always be true if the base is constant.

# Riemann Sums - Homework

For each problem, approximate the area under the given function using the specified number of rectangles/ trapezoids. You are to do all 4 methods to approximate the areas.

| # | Function | Interval | Number | Left Rectangles | Right Rectangles | Midpoint Rectangles | Trapezoids |
|---|---|---|---|---|---|---|---|
| 1 | $f(x) = x^2 - 3x + 4$ | [1,4] | 6 | 9.125 | 12.125 | 10.438 | 10.625 |
| 2 | $f(x) = \sqrt{x}$ | [2,6] | 8 | 7.650 | 8.168 | 7.914 | 7.909 |
| 3 | $f(x) = 2^x$ | [0,1] | 5 | 1.345 | 1.545 | 1.442 | 1.445 |
| 4 | $f(x) = \sin x$ | [0,π] | 8 | 1.974 | 1.974 | 2.013 | 1.974 |

5. Roger decides to run a marathon. Roger's friend Jeff rides behind him on a bicycle and clocks his pace every 15 minutes. Roger starts out strong, but after an hour and a half he is so exhausted that he has to stop. The data Jeff collected is summarized below. Assuming that Roger's speed is always decreasing, estimate the distance that Roger ran in a) the first half hour and b) the entire race. (Trapezoids)

| Time spent running (min) | 0 | 15 | 30 | 45 | 60 | 75 | 90 |
|---|---|---|---|---|---|---|---|
| Speed (mph) | 12 | 11 | 10 | 10 | 8 | 7 | 0 |

a. $\boxed{d \approx \frac{.25}{2}(12 + 22 + 10) = 5.5 \text{ miles}}$

b. $\boxed{d \approx \frac{.25}{2}(12 + 22 + 20 + 20 + 16 + 14 + 0) = 13 \text{ miles}}$

6. Coal gas is produced at a gasworks. Pollutants in the air are removed by scrubbers, which become less and less efficient as time goes on. Measurements are made at the start of each month (although some months were neglected) showing the rate at which pollutants in the gas are as follows. Use trapezoids to estimate the total number of tons of coal removed over 9 months.

| Time (months) | 0 | 1 | 3 | 4 | 6 | 7 | 9 |
|---|---|---|---|---|---|---|---|
| Rate pollutants are escaping (tons/month) | 5 | 7 | 8 | 10 | 13 | 16 | 20 |

$$\boxed{T \approx .5(5+7) + 1(7+8) + .5(8+10) + 1(10+13) + .5(13+16) + 1(16+20) = 103.5 \text{ tons}}$$

7. For $0 \le t \le 1$, a bug is crawling at a velocity $v$, determined by the formula $v = \dfrac{1}{1+t}$, where $t$ is in hours and $v$ is in meters/hr. Find the distance that the bug crawls during this hour using 10 minute increments.

$$\boxed{d \approx \frac{1}{2}\left(\frac{1}{6}\right)\left[v(0) + 2v\left(\frac{1}{6}\right) + 2v\left(\frac{2}{6}\right) + 2v\left(\frac{3}{6}\right) + 2v\left(\frac{4}{6}\right) + 2v\left(\frac{5}{6}\right) + v(1)\right] = 0.695 \text{ meters}}$$

8. An object has zero initial velocity and a constant acceleration of $32 \text{ ft/sec}^2$. Complete the chart to find the velocity at these specified times. Then determine the distance traveled in 4 seconds. $\boxed{v = 32t}$

| $t$ (sec) | 0 | .5 | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 |
|---|---|---|---|---|---|---|---|---|---|
| $v$ (ft/sec) | 0 | 16 | 32 | 48 | 64 | 80 | 96 | 112 | 128 |

$$\boxed{d \approx \frac{1}{2}\left(\frac{1}{2}\right)\left[0 + 2(16) + 2(32) + 2(48) + 2(64) + 2(80) + 2(96) + 2(112) + 128\right] = 256 \text{ feet}}$$