Finding the Exact Area Under a Curve - Classwork

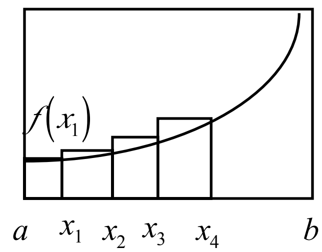
Now that we have used a finite number of right, left, and mid rectangles as well as trapezoids to approximate the area under a curve, we now extend the concept to finding the exact area under the curve by looking at *n* rectangles and calculating the limit of the sum of the areas of those rectangles as *n* approaches infinity.

In the method shown below, we will assume right rectangles. Left rectangles are possible as well but since either method will give the same answer, we will just settle on one method. This and the next page will be set up so that the general method and description will be on the left side of the page, and an example (specific problem) will be on the right side.

General Problem: Find the exact area between a curve $y = f(x)$, the *x*-axis, and $x = a$ and $x = b$. We assume the graph will be above the *x*-axis.

Specific Problem: Find the exact area between the curve $f(x) = x^2 + 1$, the *x*-axis, and $x = 1$ and $x = 3$

• Make a simple sketch of the function



• Determine Δx = the base of each rectangle

$\Delta x = \dfrac{b-a}{n}$ where *n* = the number of rectangles

$$\Delta x = \frac{3-1}{n} = \frac{2}{n}$$

• Find the *x*-coordinates of the right leg of each rectangle, $x_1, x_2, x_3, \ldots x_i$

$x_i = a + i\Delta x$

$$x_i = a + i\frac{2}{n} = 1 + \frac{2i}{n}$$

• Find the heights of the right leg of each rectangle

$f(x_1)$, $f(x_2)$, $f(x_3)$, … $f(x_i)$

The algebra can be messy.

$$\begin{aligned} f(x_i) &= f\left(1 + \frac{2i}{n}\right) \\ &= \left(1 + \frac{2i}{n}\right)^2 + 1 \\ &= 1 + \frac{4i}{n} + \frac{4i^2}{n^2} + 1 \\ &= 2 + \frac{4i}{n} + \frac{4i^2}{n^2} \end{aligned}$$

• Now we need the area of each rectangle

$A_1, A_2, A_3, \ldots A_i$

A_i = base of rectangle$_i$ • height of rectangle$_i$

The power of i is always 1 less than the power of n

$$A_i = \left(\frac{2}{n}\right)\left(2+\frac{4i}{n}+\frac{4i^2}{n^2}\right) = \frac{4}{n}+\frac{8i}{n^2}+\frac{8i^2}{n^3}$$

• Now we need to sum up all the areas

$$A = \sum_{i=1}^{n} A_i$$

$$A = \sum_{i=1}^{n}\left(\frac{4}{n}+\frac{8i}{n^2}+\frac{8i^2}{n^3}\right)$$

• Distribute the $\sum$ and factor out the constant and n's.

Everything to the right of the $\sum$ should be 1 or i's

$$A = \sum_{i=1}^{n}\frac{4}{n}+\sum_{i=1}^{n}\frac{8i}{n^2}+\sum_{i=1}^{n}\frac{8i^2}{n^3}$$

$$A = \frac{4}{n}\sum_{i=1}^{n}1 + \frac{8}{n^2}\sum_{i=1}^{n}i + \frac{8}{n^3}\sum_{i=1}^{n}i^2$$

• Now apply your $\sum$ rules

Do any canceling you can with the numbers and n

Multiply out each expression

Write every term over a denominator

Do any canceling you can

This is a formula for the area using n rectangles

$$A = \left(\frac{4}{n}\right)n + \left(\frac{8}{n^2}\right)\frac{n(n+1)}{2} + \left(\frac{8}{n^3}\right)\frac{n(n+1)(2n+1)}{6}$$

$$A = 4 + \frac{4(n+1)}{n} + \frac{4(2n^2+3n+1)}{3n^2}$$

$$A = 4 + \frac{4n}{n} + \frac{4}{n} + \frac{8n^2}{3n^2} + \frac{12n}{3n^2} + \frac{4}{3n^2}$$

$$A = 4 + 4 + \frac{4}{n} + \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}$$

• The last step: You want $\lim\limits_{n\to\infty} A$

Since $\lim\limits_{n\to\infty}\frac{1}{n} = 0$, all n's in denominator disappear

So simply add the numbers

$$A = 4 + 4 + \frac{8}{3}$$

$$A = \frac{32}{3}$$

Example 2) Find the area bordered by the curve $f(x) = 6 + x - x^2$, and the lines $x = 1$ and $x = 2$.

$$\Delta x = \frac{1}{n}$$

$$x_i = 1 + \frac{i}{n}$$

$$f(x_i) = 6 + 1 + \frac{i}{n} - \left(1+\frac{i}{n}\right)^2$$

$$f(x_i) = 7 + \frac{i}{n} - 1 - \frac{2i}{n} - \frac{i^2}{n^2}$$

$$f(x_i) = 6 - \frac{i}{n} - \frac{i^2}{n^2}$$

$$A_i = \left(\frac{1}{n}\right)\left(6 - \frac{i}{n} - \frac{i^2}{n^2}\right) = \frac{6}{n} - \frac{i}{n^2} - \frac{i^2}{n^3}$$

$$A = \sum_{i=1}^{n}\frac{6}{n} - \sum_{i=1}^{n}\frac{i}{n^2} - \sum_{i=1}^{n}\frac{i^2}{n^3}$$

$$A_n = \frac{6}{n}(n) - \frac{1}{n^2}\frac{(n)(n+1)}{2} - \frac{1}{n^3}\frac{(n)(n+1)(2n+1)}{6}$$

$$A_n = 6 - \frac{1}{2} - \frac{1}{2n} - \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$$

$$\lim_{n\to\infty} A_n = 6 - \frac{1}{2} - \frac{1}{3} = \frac{31}{6}$$

Finding the Exact Area Under a Curve - Homework

Find the area between the following curves, the *x*-axis, between the two given *x*-values:

1. $f(x) = 4x - 2,\ x = 1 \text{ and } x = 4$ Answer: 24

$$\Delta x = \frac{3}{n} \qquad x_i = 1 + \frac{3i}{n}$$
$$f(x_i) = 4\left(1 + \frac{3i}{n}\right) - 2$$
$$f(x_i) = 2 + \frac{12i}{n}$$
$$A_i = \left(\frac{3}{n}\right)\left(2 + \frac{12i}{n}\right) = \frac{6}{n} + \frac{36i}{n^2}$$

$$A = \sum_{i=1}^{n} \frac{2}{n} + \sum_{i=1}^{n} \frac{12i}{n^2}$$
$$A_n = \frac{6}{n}(n) + \frac{36}{n^2}\frac{(n)(n+1)}{2}$$
$$\lim_{n\to\infty} A_n = 24$$

2. $f(x) = x^2 + x,\ x = 2 \text{ and } x = 4$ Answer: $\frac{74}{3}$

$$\Delta x = \frac{2}{n} \qquad x_i = 2 + \frac{2i}{n}$$
$$f(x_i) = \left(2 + \frac{2i}{n}\right)^2 + 2 + \frac{2i}{n}$$
$$f(x_i) = 6 + \frac{10i}{n} + \frac{4i^2}{n^2}$$

$$A_i = \left(\frac{2}{n}\right)\left(6 + \frac{10i}{n} + \frac{4i^2}{n^2}\right)$$
$$A_i = \frac{12}{n} + \frac{20i}{n^2} + \frac{8i^2}{n^3}$$
$$\lim_{n\to\infty} A_n = 12 + 10 + \frac{8}{3} = \frac{74}{3}$$

3. $f(x) = 2x^2 - 3x + 1,\ x = 1 \text{ and } x = 2$ Answer: $\frac{7}{6}$

$$\Delta x = \frac{1}{n} \qquad x_i = 1 + \frac{i}{n}$$
$$f(x_i) = 2\left(1 + \frac{i}{n}\right)^2 - 3\left(1 + \frac{i}{n}\right) + 1$$
$$f(x_i) = \frac{i}{n} + \frac{2i^2}{n^2}$$

$$A_i = \left(\frac{1}{n}\right)\left(\frac{i}{n} + \frac{2i^2}{n^2}\right)$$
$$A_i = \frac{i}{n^2} + \frac{2i^2}{n^3}$$
$$\lim_{n\to\infty} A_n = \frac{1}{2} + \frac{2}{3} = \frac{7}{6}$$

4. $f(x) = 9 - x^2$ and the x - intercepts Answer: 36

$$\Delta x = \frac{3}{n} \qquad x_i = 0 + \frac{3i}{n}$$
$$f(x_i) = 9 - \left(\frac{3i}{n}\right)^2$$
$$f(x_i) = 9 - \frac{9i^2}{n}$$

$$A_i = \left(\frac{3}{n}\right)\left(9 - \frac{9i^2}{n}\right)$$
$$A_i = \frac{27}{n} - \frac{27i^2}{n^2}$$
$$\lim_{n\to\infty} A_n = 2\left(27 - \frac{54}{6}\right) = 36$$

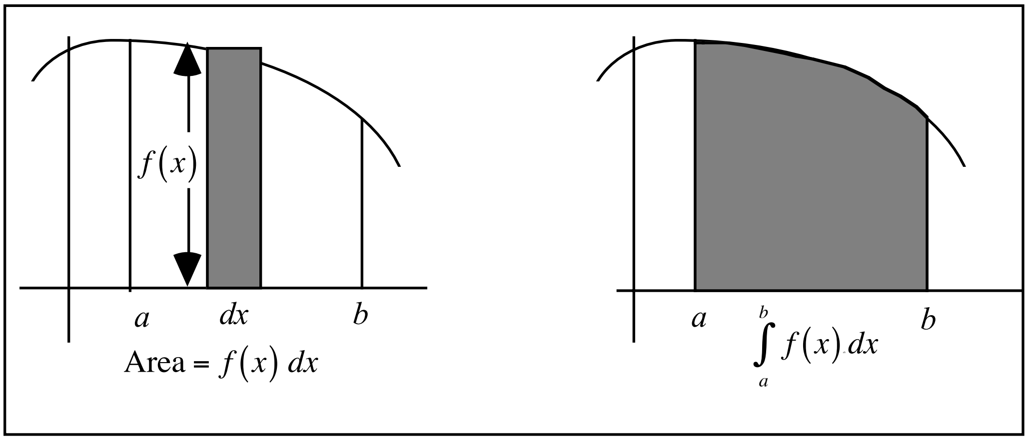
5. $f(x) = x^3,\ x = 0 \text{ and } x = 4$ Answer: 64

$$\Delta x = \frac{4}{n} \qquad x_i = 0 + \frac{4i}{n}$$
$$f(x_i) = \left(\frac{4i}{n}\right)^3$$
$$f(x_i) = \frac{64i^3}{n^3}$$

$$A_i = \left(\frac{4}{n}\right)\left(\frac{64i^3}{n^3}\right)$$
$$A_i = \frac{256i^3}{n^4}$$
$$\lim_{n\to\infty} A_n = \frac{256}{n^4}\left(\frac{n^2(n+1^2)}{4}\right) = 64$$

The Definite Integral as Area - Classwork

Instead of using the expression "the area under the curve $f(x)$ between $x = a$ and $x = b$, we will now denote a shorthand to represent the same thing. We will use what is called "a definite integral." The definite integral sign is the same as the indefinite integral sign ($\int$) but will contain two limits of integration. The form is as follows: $\int_a^b f(x)\,dx$. While this does not seems to make much sense, there is a reason for it. The $f(x)$ represents the height of any one rectangle while the dx represents the width of any one rectangle. So $f(x)\,dx$ means the area of any one rectangle. The integral represents the sum of an infinite number of these rectangles. The *a* represents the starting place for these rectangles while the *b* represents the ending place for these integrals.



The area of one rectangle = $f(x)\,dx$

The sum of an infinite number of rectangle areas. Each rectangle is infinitely thin.

When $a < b$, we are determining the area under the curve from left to right. In that case, our dx is a positive number. If $f(x)$ is above the axis, then $\int_a^b f(x)\,dx$ will be a positive number.

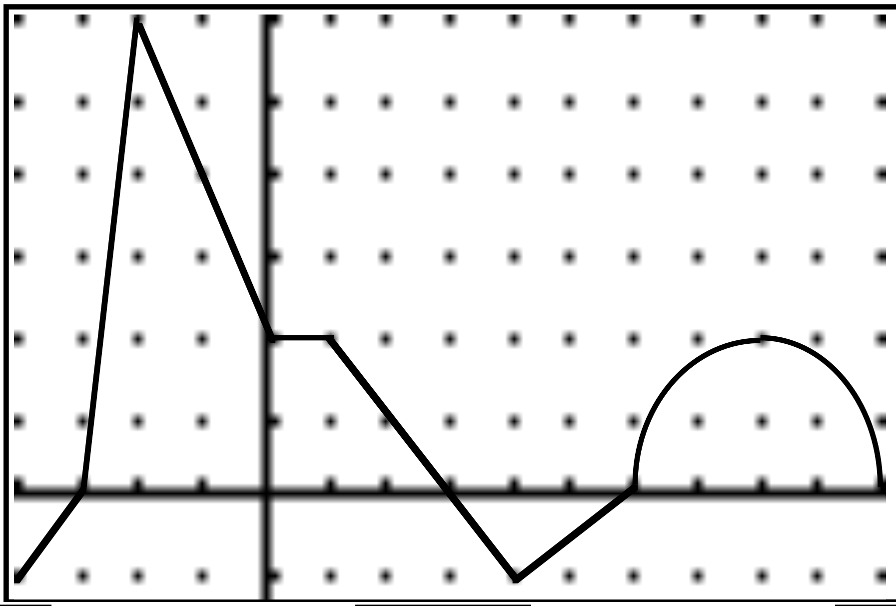
When $b < a$, we are determining the area under the curve from right to left. In that case, our dx is a negative number. If $f(x)$ is above the axis, then $\int_a^b f(x)\,dx$ will be a negative number. This can be summarized below:

	$f(x) > 0$ (curve above axis)	$f(x) < 0$ (curve below axis)
$dx > 0$ (left to right) ($a < b$)	$\int_a^b f(x)\,dx > 0$ (Area positive)	$\int_a^b f(x)\,dx < 0$ (Area negative)
$dx < 0$ (right to left) ($b < a$)	$\int_a^b f(x)\,dx < 0$ (Area negative)	$\int_a^b f(x)\,dx > 0$ (Area positive)

Furthermore, there are three more rules which will make sense to you:

1. $\int_a^a f(x)\,dx = 0$ - If we start at *a* and end at *a*, there is no area.
2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ - From *a* to *b* gives an area. From *b* to *a* gives the negative of this area.
3. $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$ - Total the area from *a* to *b*, add area from *b* to *c* = the area from *a* to *c*.

Example) Below you are given the graph of $f(x)$ formed by lines and a semi-circle. Find the definite integrals.



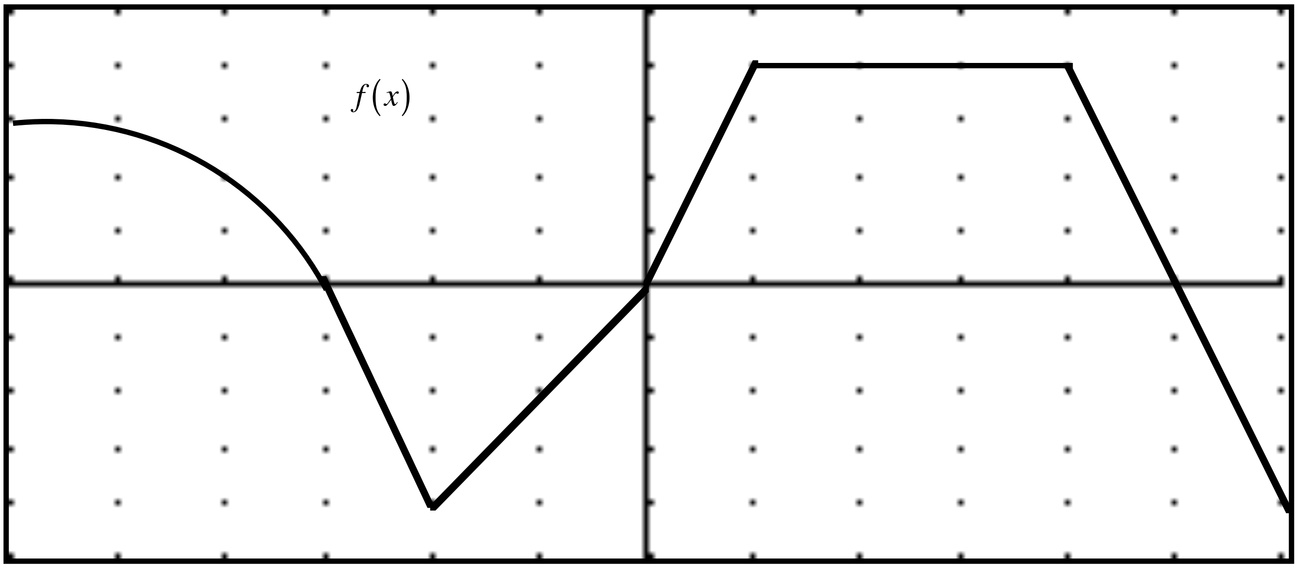
This is the graph of $f(x)$

1. $\int_{4}^{4} f(t)\,dt = 0$
2. $\int_{0}^{1} f(t)\,dt = 2$
3. $\int_{1}^{3} f(t)\,dt = 2$
4. $\int_{0}^{3} f(t)\,dt = 4$
5. $\int_{3}^{6} f(t)\,dt = -1.5$
6. $\int_{6}^{3} f(t)\,dt = 1.5$
7. $\int_{0}^{6} f(t)\,dt = 2.5$
8. $\int_{6}^{10} f(t)\,dt = 2\pi$
9. $\int_{10}^{6} f(t)\,dt = -2\pi$
10. $\int_{0}^{10} f(t)\,dt = 2.5 + 2\pi$
11. $\int_{10}^{0} f(t)\,dt = -2.5 - 2\pi$
12. $\int_{-1}^{0} f(t)\,dt = 3$
13. $\int_{-3}^{0} f(t)\,dt = 11$
14. $\int_{0}^{-3} f(t)\,dt = -11$
15. $\int_{-4}^{-3} f(t)\,dt = -.5$
16. $\int_{-4}^{0} f(t)\,dt = 10.5$
17. $\int_{-4}^{10} f(t)\,dt = 13 + 2\pi$
18. $\left|\int_{0}^{10} f(t)\,dt\right| = 2.5 + 2\pi$
19. $\int_{0}^{10} |f(t)|\,dt = 5.5 + 2\pi$
20. $\left|\int_{-4}^{10} f(t)\,dt\right| = 13 + 2\pi$
21. $\int_{10}^{-4} |\,2f(t)\,|\,dt = -34 - 4\pi$

Suppose $\int_{-2}^{5} f(x)\,dx = 18$, $\int_{-2}^{5} g(x)\,dx = 5$, $\int_{-2}^{5} h(x)\,dx = -11$ and $\int_{-2}^{8} f(x)\,dx = 0$, find

22. $\int_{-2}^{5} \left(f(x) + g(x)\right) dx = 23$
23. $\int_{-2}^{5} \left[f(x) + g(x) - h(x)\right] dx = 34$
24. $\int_{5}^{-2} 4g(x)\,dx = -20$
25. $\int_{-2}^{5} \left(g(x) + 2\right) dx = 19$
26. $\int_{-2}^{5} \left(f(x) - 6\right) dx = -24$
27. $\int_{0}^{7} h(x-2)\,dx = -11$
28. $\int_{-4}^{3} g(x+2)\,dx = 5$
29. $\int_{5}^{8} f(x)\,dx = -18$
30. $\int_{1}^{8} \left[f(x-3) + 3\right] dx = 39$

The Definite Integral as Area - Homework



1. $\int_0^1 f(x)\,dx = 2$

2. $\int_2^4 f(x)\,dx = 8$

3. $\int_1^4 f(x)\,dx = 12$

4. $\int_5^5 f(x)\,dx = 0$

5. $\int_4^5 f(x)\,dx = 2$

6. $\int_5^6 f(x)\,dx = -2$

7. $\int_4^6 f(x)\,dx = 0$

8. $\int_0^6 f(x)\,dx = 14$

9. $\int_3^2 f(x)\,dx = -4$

10. $\int_5^0 f(x)\,dx = -16$

11. $\int_6^0 f(x)\,dx = -14$

12. $\int_{-3}^0 f(x)\,dx = -6$

13. $\int_0^{-3} f(x)\,dx = 6$

14. $\int_{-3}^2 f(x)\,dx = 0$

15. $\int_4^{-3} f(x)\,dx = -8$

16. $\int_{-6}^{-3} f(x)\,dx = \frac{9\pi}{4}$

17. $\int_{-3}^{-6} f(x)\,dx = \frac{-9\pi}{4}$

18. $\int_{-6}^{0} f(x)\,dx = \frac{9\pi}{4} - 6$

19. $\int_{-6}^{6} f(x)\,dx = \frac{9\pi}{4} + 8$

20. $\int_{6}^{-6} f(x)\,dx = -\frac{9\pi}{4} - 8$

21. $\left|\int_{-2}^{1} f(x)\,dx\right| = 2$

22. $\int_{-2}^{1} |f(x)|\,dx = 6$

23. $\int_{-2}^{1} |-f(x)|\,dx = 6$

24. $\int_{-6}^{6} |f(x)|\,dx = \frac{9\pi}{4} + 24$

Suppose that $\int_0^2 f(x)\,dx = 2$, $\int_1^2 f(x)\,dx = -1$, $\int_2^4 f(x)\,dx = 7$, evaluate the following:

25. $\int_1^4 f(x)\,dx = 6$

26. $\int_0^4 3f(x)\,dx = 27$

27. $\int_0^1 f(x)\,dx = 3$

28. $\int_0^1 f(x+1)\,dx = -1$

29. $\int_0^2 \left(f(x)+3\right)dx = 8$

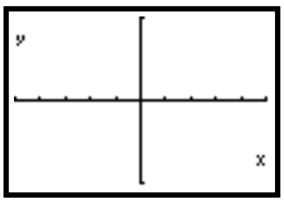
30. $\int_2^4 f(x-2)\,dx = 2$

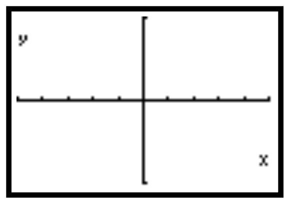
31. If $\int_0^3 f(x)\,dx = -1$, find $\int_{-3}^3 f(x)\,dx$ if f is a) even $\boxed{-2}$ b) odd $\boxed{0}$

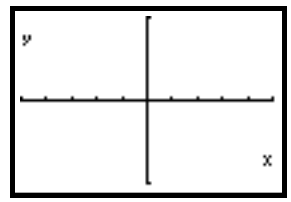
The problems on this page and the next should be done <u>without calculators.</u>

32. For each function below, use a sketch of f to evaluate the 4 definite integrals on top given the statement on the left. (a is a positive constant)

	$\int_0^2 f(x)\,dx$	$\int_1^4 f(x)\,dx$	$\int_5^1 f(x)\,dx$	$\left\lvert\int_9^{-9} f(x)\,dx\right\rvert$
$f(x)=3$	6	9	-12	54
$f(x)=a$	$2a$	$3a$	$-4a$	$18a$
$f(x)=-\sqrt{a}$	$-2\sqrt{a}$	$-3\sqrt{a}$	$4\sqrt{a}$	$18\sqrt{a}$

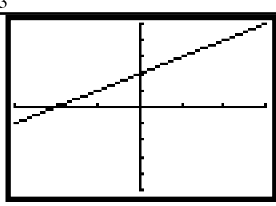




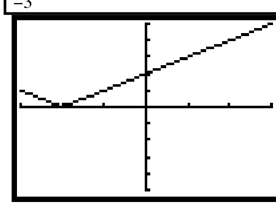


33. Evaluate the following by making a sketch of the function.

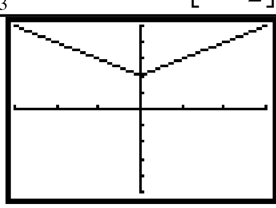
a. $\int_{-3}^{3} (x+2)\,dx = \frac{-1}{2}+\frac{25}{2}=12$



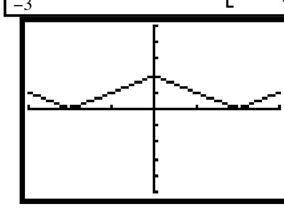
b. $\int_{-3}^{3} |x+2|\,dx = \frac{1}{2}+\frac{25}{2}=13$



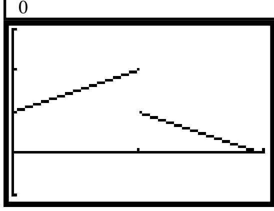
c. $\int_{-3}^{3} \left||x|+2\right|\,dx = 2\left[6+\frac{9}{2}\right]=21$



d. $\int_{-3}^{3} \left|\,2-|x|\,\right|\,dx = 2\left[2+\frac{1}{2}\right]=5$

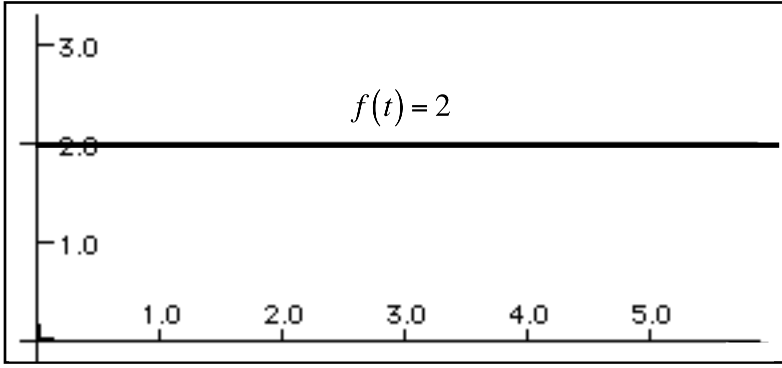


34. Evaluate $\int_0^2 f(x)\,dx$ where $f(x)=\begin{cases}1+x, & \text{if } 0 \le x \le 1\\ 2-x, & \text{if } 1 < x \le 2\end{cases} = 2$. Make a sketch of the function.



The Accumulation Function - Classwork

Now that we understand that the concept of a definite integral is nothing more than an area, let us consider a very special type of area problem. Suppose we are given a function $f(x)=2$. We know that to be a horizontal line at $y = 2$. First, realize that the equation of the graph of $f(x)=2$ is the same as $f(t)=2$ is the same as $f(k)=2$. Whether we use x, t, or k, it does not matter. The graph is still a horizontal line. We are used to using x but we will see a good reason that we will occasionally be using another letter.



Consider the expression $\int_0^x f(t)\,dt$

Above, we have the graph $f(t)=2$. We now want to consider the expression $\int_0^x f(t)\,dt$. What is this? It is a function of x. As x changes, $\int_0^x f(t)\,dt$ changes as well. Complete the chart below.

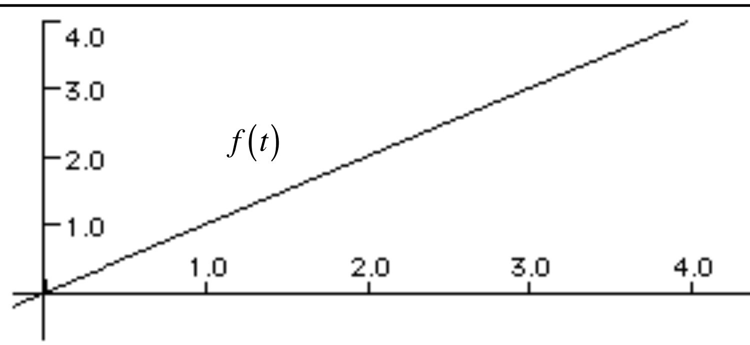
x	0	1	2	3	4	5
$\int_0^x f(t)\,dt$	**0**	**2**	**4**	**6**	**8**	**10**

It should be apparent that as x gets bigger, $\int_0^x f(t)\,dt$ increases as well. What we appear to be doing is "accumulating area" and we call $\int_0^x f(t)\,dt$ <u>the accumulation function</u>. Finally, it should be obvious why we use $f(t)=2$ to describe our line rather than $f(x)=2$. $\int_0^x f(x)\,dx$ would be confusing.

Let' s calculate the value of

$\int_0^x f(t)\,dt$ for $x=0$ to 4

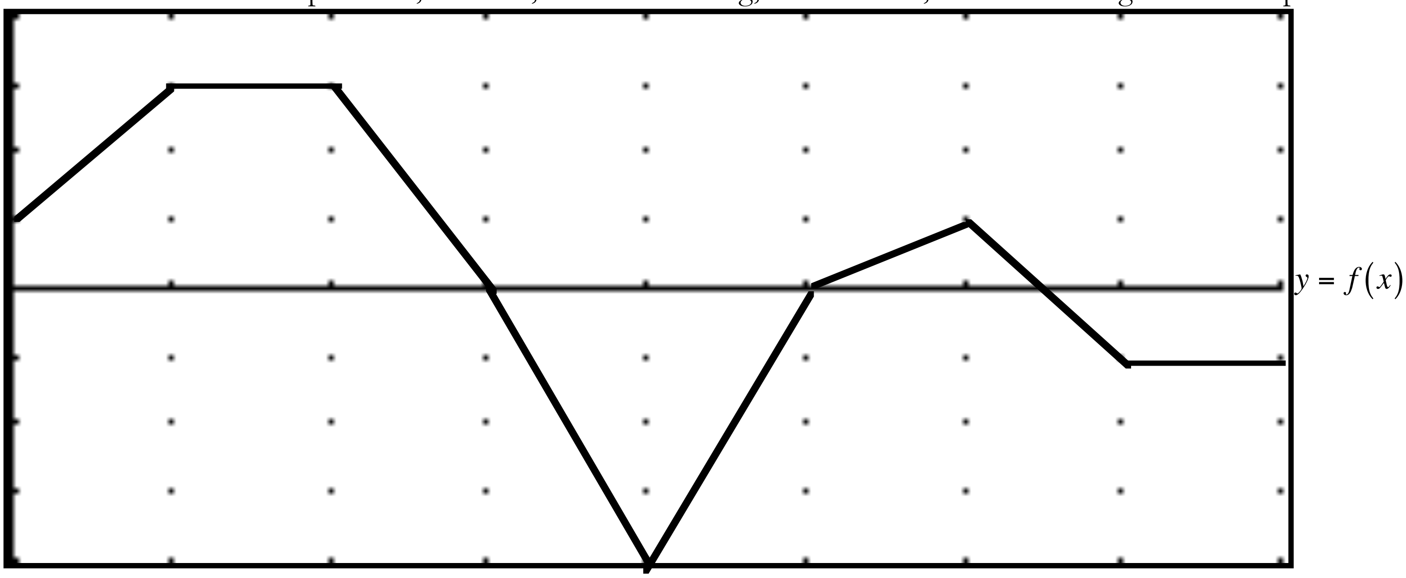
Complete the chart below



x	0	1	2	3	4
$\int_0^x f(t)\,dt$	**0**	**.5**	**2**	**4.5**	**8**

Example 1) Let $F(x) = \int_0^x f(t)\,dt$ where the graph of $f(x)$ is below. Remember $f(x)$ is the same thing as $f(t)$.

Think of $f(x)$ as the <u>rate</u> of snowfall over a period of time. For instance, at x = 1, snow is falling at 3 inches per hour, at x = 3, it is not snowing, and at x = 4, snow is melting at 4 inches per hour.



a. Complete the chart. In the snow analogy, $F(x)$ represents the accumulation of snow over time.

x	0	1	2	3	4	5	6	6.5	7	8
$F(x)$	**0**	**2**	**5**	**6.5**	**4.5**	**2.5**	**3**	**3.25**	**3**	**2**

b. Now let's consider $F'(x) = \frac{d}{dx}\int_0^x f(t)\,dt$. If we take the derivative of an integral, what would you expect to happen? cancellation. So $F'(x) = \frac{d}{dx}\int_0^x f(t)\,dt$ is the same thing as $f(x)$.

Knowing that, let's complete the chart.

x	0	1	2	3	4	5	6	6.5	7	8
$F'(x)$	**1**	**3**	**3**	**0**	**-4**	**0**	**1**	**0**	**-1**	**-1**

c. On what subintervals of [0, 8] is F increasing? $(0,3)\cup(5,6.5)$ Decreasing? $(3,5)\cup(6.5,8)$

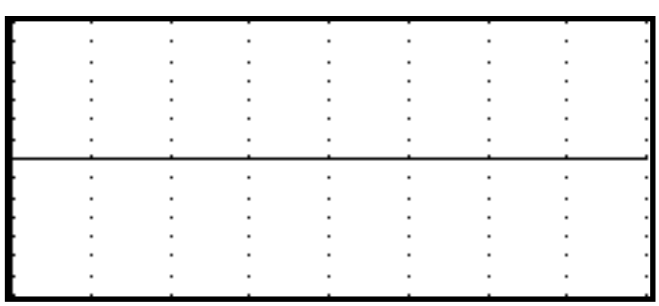
d. Where in the interval [0, 8] does F achieve its minimum and maximum value? What are those values?

Max = 6.5 at x = 3 Min = 0 at x = 0

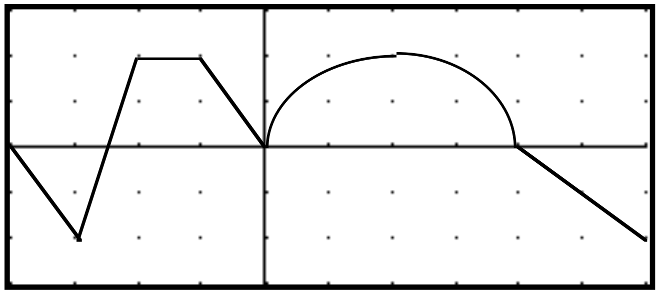
f. Find the concavity of F and any inflection points. Justify your answer

Concave up $(0,1)\cup(4,6)$, Concave down $(2,4)\cup(6,7)$, Inflection pts: $x = 4, x = 6$

h. Sketch a rough graph of $F(x)$



Example 2) Let $F(x) = \int_0^x f(t)\,dt$ where f is the function graphed below (consisting of lines and a semi-circle)



$y = f(x)$

Find the following:

a) $F(0) = 0$ b) $F(2) = \pi$ c) $F(4) = 2\pi$ d) $F(6) = 2\pi - 2$

e) $F(-1) = -1$ f) $F(-2) = -3$ g) $F(-3) = -3$ h) $F(-4) = -2$

i) $F'(4) = 0$ j) $F'(2) = 2$ k) $F'(6) = -2$ l) $F'(-3) = -2$

m) On what subintervals of $[-4,6]$ is F increasing and decreasing. Justify your answer.

increasing where $F' > 0$ $(-2.5,0)\cup(0,4)$ decreasing where $F' < 0$ $(-4,-2,5)\cup(4,6)$

n) Where in the interval $[-4,6]$ does F achieve its minimum value? What is the minimum value?

-3.5 at $x = -2.5$

o) Where in the interval $[-4,6]$ does F achieve its maximum value? What is the minimum value?

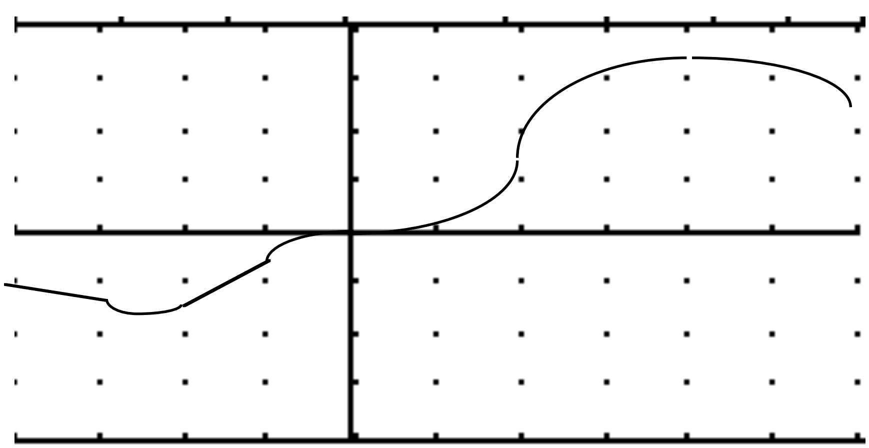
2π at $x = 4$

p) Where on the interval $[-4,6]$ is F concave up? Concave down? Justify your answer.

Concave up where F' is increasing: $(-3,-2)\cup(0,2)$, Down where F' is decreasing $(-4,-3)\cup(-1,0)\cup(2,6)$

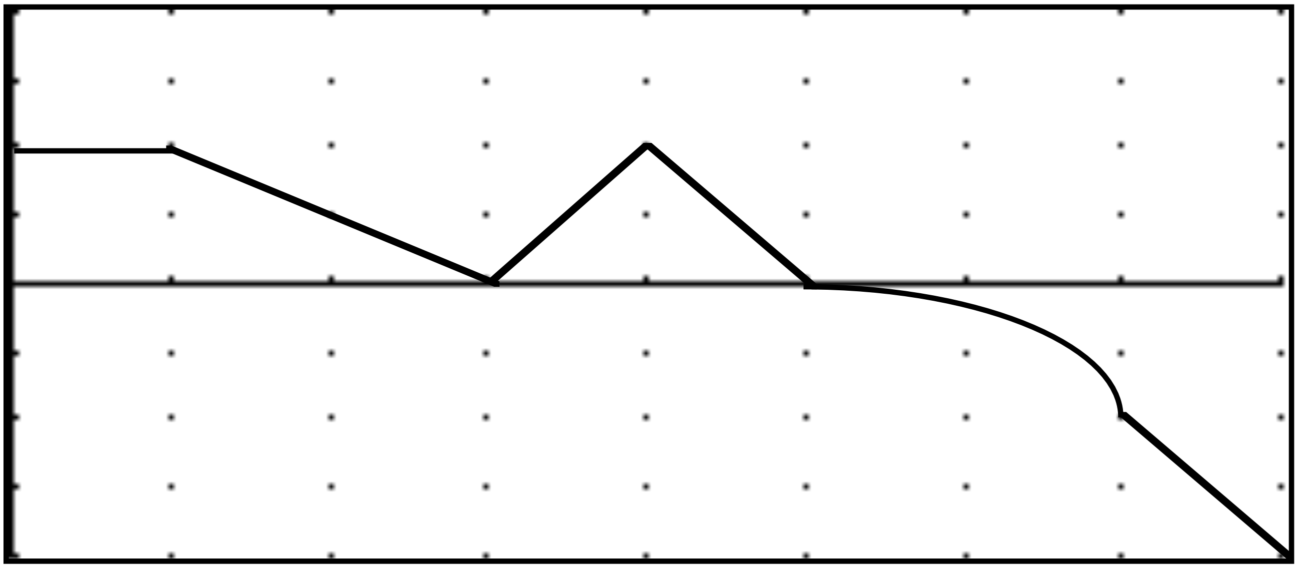
q) Where does F have points of inflection? $x = -3, x = 0, x = 2$

r) Sketch the function F.



Assume each tic mark on the y-axis is 2 units

The Accumulation Function - Homework



1. Let $F(x) = \int_0^x f(t)\,dt$ where the graph of $f(x)$ is above (the graph consists of lines and a quarter circle)

a. Complete the chart

x	0	1	2	3	4	5	7	8
$F(x)$	0	2	3.5	4	5	6	$2+\pi$	$\pi-1$
$F'(x)$	**2**	**2**	**1**	**0**	**2**	**0**	**-2**	**-4**

b. On what subintervals of [0, 8] is F increasing? Decreasing? Justify your answer.

Increasing where $F' > 0 : (0,3) \cup (3,5)$ Decreasing where $F' < 0 : (5,8)$

c. Where in the interval [0, 8] does F achieve its minimum value? What is the minimum value? Justify answer.

0 at $x = 0$: Positive area is accumulated until $x = 5$ and then negative area until $x = 8$. There is more positive area than negative area.

d. Where in the interval [0, 8] does F achieve its maximum value? What is the maximum value? Justify answer.

6 at $x = 5$: Same justification as above.

e. Find the concavity of F and any inflection points. Justify answers.

Up where F' increasing : $(3,4)$ Down where F' decreasing : $(1,3) \cup (4,8)$

f. Sketch a rough graph of $F(x)$