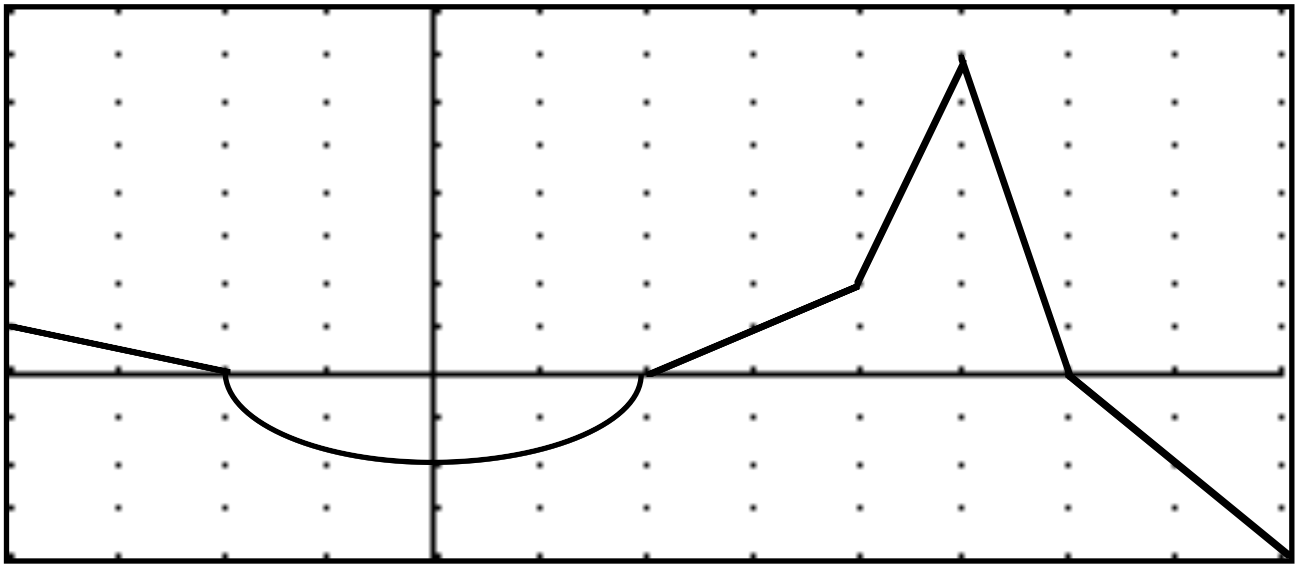


2. Let $F(x) = \int_0^x f(t)\,dt$ where the graph of $f(x)$ is above (the graph consists of lines and a semi-circle)

a. Complete the chart

x	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
$F(x)$	π-1	π-.25	π	1.8	0	-1.8	-π	-π+.5	2-π	6.5-π	10-π	9-π	6-π
$F'(x)$	**1**	**.5**	**0**	**-1.8**	**-2**	**-1.8**	**0**	**1**	**2**	**7**	**0**	**-2**	**-4**

b. On what subintervals of $[-4,8]$ is F increasing? Decreasing?

Increasing where : $F' > 0 : (-4,-2) \cup (2,6)$ Decreasing where $F' < 0 : (-2,2) \cup (6,8)$

c. Where in the interval $[-4,8]$ does F achieve its minimum value? What is the minimum value? Justify answer.

Relative minimum at $x = 2$ as $F' < 0$ if $x < 2$, $F' > 0$ if $x > 2$

$F(2) = -\pi$, $F(-4) = \pi - 1$, $F(8) = 6 - \pi$. Abs. Min $= -\pi$ at $x = 2$.

d. Where in the interval $[-4,8]$ does F achieve its maximum value? What is the maximum value? Justify answer.

Relative maximum at $x = -2$ as $F' > 0$ if $x < -2$, $F' < 0$ if $x > -2$

Relative maximum at $x = 6$ as $F' > 0$ if $x < 6$, $F' < 0$ if $x > 6$

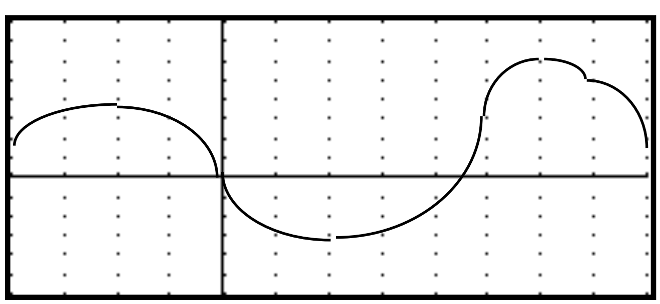
$F(-2) = \pi$, $F(6) = 10 - \pi$, $F(-4) = \pi - 1$, $F(8) = 6 - \pi$. Abs. Max $= 10 - \pi$ at $x = 6$.

e. On what subintervals of $[-4,8]$ is F concave up and concave down? Find its inflection points. Justify answers.

Concave up when F' increasing : $(0,5)$

Concave down when F' decreasing : $(-4,0) \cup (5,8)$

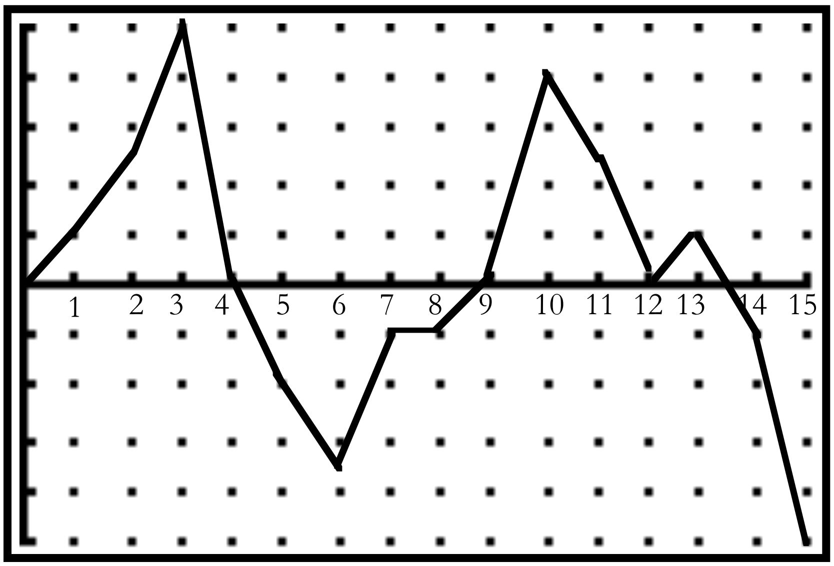
f. Sketch a rough graph of $F(x)$



The Accumulation Function - Life Application

You have bought 100 shares of XYZ stock and decide to keep it for 15 days. Below is a graph that represents the change of the price of the your stock on each day. For instance, on the end of day 1, the stock has increased by $1 a share. At the end of day 4, it has not changed. At the end of day 6, it has gone down by $3.50 a share.

We will call the graph you see below $f(t)$. made up of straight lines. It is obviously a function of time. Remember that the function represents the *change* in the value of your stock, **not** the value of the stock.



Let $F(x) = \int_0^x f(t)\,dt$. Answer the following questions:

1. Complete the chart below.

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$f(x)$	**0**	**1**	**2.5**	**5**	**0**	**-2**	**-3.5**	**-1**	**-1**	**0**	**4**	**2.5**	**0**	**1**	**-1**	**-5**
$F(x)$	**0**	**.5**	**2.25**	**6**	**8.5**	**7.5**	**4.75**	**2.5**	**1.5**	**1**	**3**	**6.25**	**7.5**	**8**	**8**	**5**

2. What is the real life meaning of $F(x)$?

Accumulated value of stock (assuming it is changing steadily) from day 0 to day x.

3. For what intervals is $F(x)$ increasing and decreasing. Justify your answer.

Increasing where $F' > 0 : (0,4) \cup (9,12) \cup (12,13.5)$: Decreasing where : $F' < 0 : (4,9) \cup (13.5,15)$

4. In the interval [0, 15], find the minimum value of $F(x)$ and the day it is reached. Justify your answer.

Potential minimums: $x = 0, x = 9, x = 15$. Based on the chart above, the minimum value of $F(x)$ is $0 on da

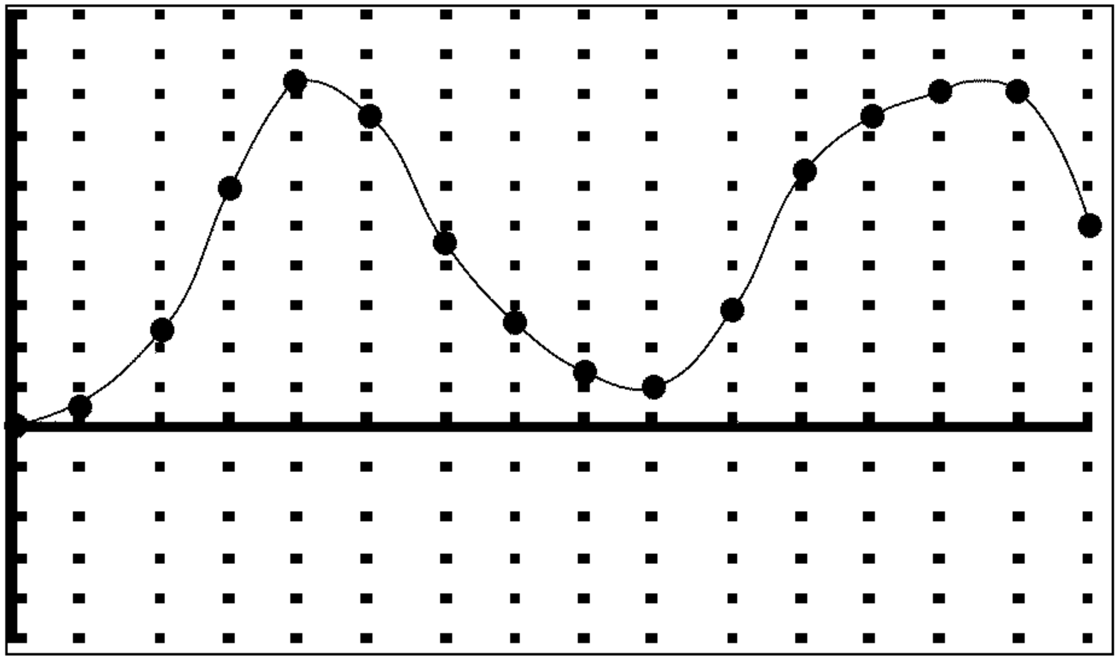
5. In the interval [0, 15], find the maximum value of $F(x)$ and the day it is reached. Justify your answer.

Potential miximums: $x = 4, x = 13$. Based on the chart above, the minimum value of $F(x)$ is $8.50 on day

6. Determine the concavity of $F(x)$ and justify your answer.

Up: F' increasing - $(0,3) \cup (6,7) \cup (8,10) \cup (12,13)$ Down: F' decreasing - $(3,6) \cup (10,12) \cup (13,15)$

7. Below, sketch $F(x)$.



8) Based on your findings, find:

a) how much money you made(lost) on XYZ stock in that 15 day time period.

$500

b) the day that the stock's value had the biggest rise

day 3

c) the days between which the stock's value had the steepest rise (not the same question)

between day 9 and day 10

d) the day that the stock's value had the biggest decline

day 15

e) the days between which the stock's value had the steepest decline (not the same question)

between day 3 and day 4

f) the day you wished you sold your stock

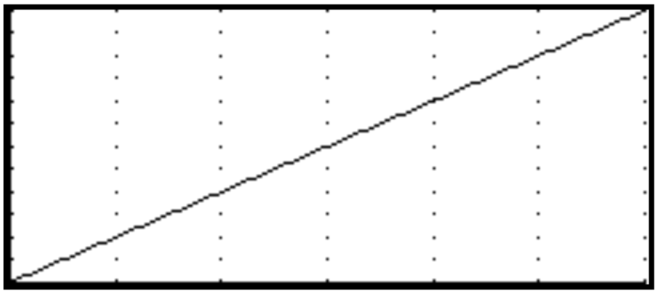
day 4

g) the day you are glad you didn't sell your stock

day 9

The Fundamental Theorem of Calculus - Classwork

Example 1) On your calculator, graph the function $y = 2x$ on the interval [0, 6]. You should get the graph below.



Now let

$$F(x) = \int_0^x 2t\,dt$$

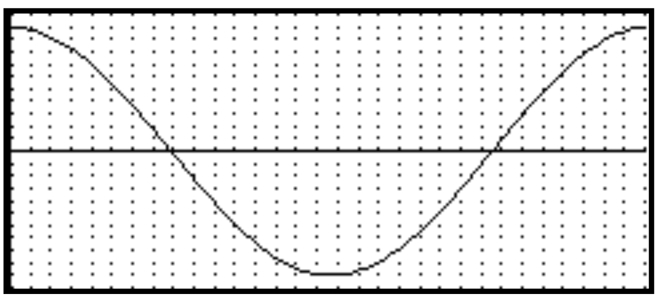
This is the accumulated area under the graph of the function to the left. Complete the chart below.

x	0	1	2	3	4	5	6
$F(x)$	**0**	**1**	**4**	**9**	**16**	**25**	**36**

What is the relationship that you see between x and $F(x)$? squares

So it appears that $\int_0^x 2t\,dt = x^2$

Example 2) On your calculator, graph $y = \cos x$ on the interval [0, 2π]. You should get the graph below.



Now let

$$F(x) = \int_0^x \cos t\,dt$$

This is the accumulated area under the curve $y = \cos x$

On your calculator, set your accuracy to 2 decimal places and define Y2 as FnInt(Y1,X,0,X). FnInt is located in Math 9. Then go to 2nd TBLSET, define TblStart = 0 and ΔTbl= .2. Then press 2nd Table. Your accumulated area should be located in Y2.

b. We need to get this set of values in to a list. Press STAT EDIT L1 can be defined as 2nd List Seq(X,X,0,6.2,.2). L2 is defined as Y2(L1). It will take a few seconds to compute these values. Now go to Stat Plot, turn Plot1 on, and perform a scatterplot with L1 vs. L2

c. Go to Y= and turn both Y1 and Y2 off. Now graph the function.

d. Guess the function whose scatterplot you have $\sin x$ So it appears that $\int_0^x \cos t\,dt = \sin x$

We are now ready to make the connection to area under a curve and indefinite integration introduced earlier. This is accomplished through the Fundamental Theorem of Calculus (Newton and Leibniz).

The Fundamental Theorem of Calculus

If a function f is continuous on the closed interval $[a,b]$, and F is an antiderivative of f on $[a,b]$, then $\int_a^b f(x)\,dx = F(b) - F(a)$

What the Fundamental Theorem of Calculus (FTC) allows you to do is to find the area under the curve by the process of integration and not having to use the limit of a sum.

Examples) Find the value of the definite integrals below.

1) $\int_1^2 x^2 dx$

$\left.\frac{x^3}{3}\right|_1^2$

$\frac{7}{3}$

2) $\int_{-1}^3 (3x^2 - 2x - 1)\,dx$

$\left.x^3 - x^2 - x\right|_{-1}^3$

16

3) $\int_0^9 \sqrt{x}\,dx$

$\left.\frac{2x^{3/2}}{3}\right|_0^9$

18

4) $\int_5^{10} \frac{2}{3x^2}\,dx$

$\left.\frac{-2}{3x}\right|_{.5}^{10}$

$\frac{1}{15}$

5) $\int_{-1}^1 (2x-1)^2\,dx$

$\left.\frac{4x^3}{3} - 2x^2 + x\right|_{-1}^1$

$\frac{14}{3}$

6) $\int_1^8 \frac{x-2}{\sqrt[3]{x}}\,dx = \int_1^8 x^{2/3} - 2x^{-1/3}\,dx$

$\left.\frac{3x^{5/3}}{5} - 3x^{2/3} + x\right|_1^8$

$\frac{48}{5}$

7) $\int_0^\pi (2 + \sin x)\,dx$

$\left.2x - \cos x\right|_0^\pi$

$2\pi + 2$

8) $\int_0^{\pi/4} (4x + \sec^2 x)\,dx$

$\left.2x^2 + \tan x\right|_0^{\pi/4}$

$\frac{\pi^2}{8} + 1$

9) $\int_0^5 |4 - 2x|\,dx$

$\int_0^2 (4-2x)\,dx - \int_2^5 (4-2x)\,dx$

$\left.(4x - 2x^2)\right|_0^2 - \left.(4x - 2x^2)\right|_2^5$

$4 - (-9) = 13$

Technology: Your TI-84 calculators allow you to find the value of a definite integral. The command is FnInt and is located as #9 in the Math menu. The syntax of the statement is FnInt(expression in X, X, lower, upper).

For instance, example # 1 above ... $\int_1^2 x^2 dx$ would be expressed to the calculator as FnInt(X^2, X,1,2) yielding 2.333. You can also put your expression in Y1 and your statement would be FnInt(Y1, X,1,2). The calculator is finding the integral but not by the Fundamental Theorem of Calculus. It is merely performing the summation of rectangles or trapezoids many many times.

The Fundamental Theorem of Calculus - Homework

Find the value of the definite integrals below. Confirm using your calculator.

1. $\int_0^1 3x\,dx$

$\left.\frac{3x^2}{2}\right|_0^1 = \frac{3}{2}$

2. $\int_{-2}^{3} (x-5)\,dx$

$\left[\frac{x^2}{2} - 5x\right]_{-2}^{3}$

$\frac{9}{2} - 10 - (2+10) = \frac{-45}{2}$

3. $\int_{-1}^{4} \left(x^2+2x-1\right)dx$

$\left[\frac{x^3}{3} + x^2 - x\right]_{-1}^{4}$

$\frac{64}{3} + 16 - 4 - \left(\frac{-1}{3} + 1 + 1\right) = \frac{95}{3}$

4. $\int_0^2 (2x-5)^2\,dx$

$\left[\frac{4x^3}{3} - 10x^2 + 25x\right]_0^2$

$\frac{32}{3} - 40 + 50 - (0) = \frac{62}{3}$

5. $\int_2^3 \left(\frac{4}{x^2}+1\right)dx$

$\left[\frac{-4}{x} + x\right]_2^3$

$\frac{-4}{3} + 3 - (-2+2) = \frac{5}{3}$

6. $\int_{-2}^{-1} \left(x - \frac{1}{x^2}\right)dx$

$\left[\frac{x^2}{2} + \frac{1}{x}\right]_{-2}^{-1}$

$\frac{1}{2} - 1 - \left(2 - \frac{1}{2}\right) = -2$

7. $\int_1^9 \frac{x-2}{\sqrt{x}}\,dx$

$\left[\frac{2x^{3/2}}{3} - 4x^{1/2}\right]_1^9$

$18 - 12 - \left(\frac{2}{3} - 4\right) = \frac{28}{3}$

8. $\int_{-2}^{2} \sqrt[3]{x}\,dx$

$\left[\frac{3x^{3/2}}{4}\right]_{-2}^{2} = \frac{3}{2}\sqrt{2} - \frac{3}{2}\sqrt{2} = 0$

9. $\int_0^1 \left(t^{2/3} - t^{1/3}\right)dt$

$\left[\frac{3x^{5/3}}{5} - \frac{3x^{4/3}}{4}\right]_0^1 = \frac{3}{5} - \frac{3}{4} = \frac{-3}{20}$

10. $\int_0^3 |x-2|\,dx$

$\int_0^2 (2-x)\,dx + \int_2^3 (x-2)\,dx$

$\left[2x - \frac{x^2}{2}\right]_0^2 + \left[\frac{x^2}{2} - 2x\right]_2^3$

$4 - 2 + \frac{9}{2} - 6 - (2-4) = \frac{5}{2}$

11. $\int_{-\pi/2}^{\pi/2} \cos x\,dx$

$[\sin x]_{-\pi/2}^{\pi/2} = \sin\frac{\pi}{2} - \sin\frac{-\pi}{2}$

$1 - (-1) = 2$

12. $\int_0^{\pi} (2x - \sin x)\,dx$

$\left[x^2 + \cos x\right]_0^{\pi}$

$\pi^2 - 1 - (0+1) = \pi^2 - 2$

13. $\int_0^{\pi/2} (3\sin x - 2\cos x)\,dx$

$[-3\cos x - 2\sin x]_0^{\pi/2}$

$0 - 2 - (-3-0) = 1$

14. $\int_0^{\pi/4} \left(x - \sec^2 x\right)dx$

$\left[\frac{x^2}{2} - \tan x\right]_0^{\pi/4}$

$\frac{\pi^2}{32} - 1 - (0-0) = \frac{\pi^2}{32} - 1$

15. $\int_0^{\pi/3} \sec\theta\tan\theta\,d\theta$

$[\sec\theta]_0^{\pi/3}$

$2 - 1 = 1$

Definite Integration with u-Substitution - Classwork

When you have to find a definite integral involving *u-substitution*, it is often convenient to determine the limits of integration in terms of the variable *u*, rather than having to integrate and switch back to *x* and then substitute. This is called "changing the limits."

Example 1) $\int_0^2 x(x^2+1)^2\,dx$ Start off by finding u $\boxed{u = x^2+1}$ $x = 2$, $\boxed{u=5}$

$\frac{1}{2}\int_0^2 2x(x^2+1)^2\,dx$ $\boxed{du = 2x\,dx}$ Do we have it? $x = 0$, $\boxed{u=1}$

Now write everything in terms of *u* and calculate. It is no longer necessary to switch back to *x*.

$$\boxed{\frac{1}{2}\int_1^5 u^2 du = \frac{1}{2}\frac{u^3}{3}\Big|_1^5 = \frac{125}{6} - \frac{1}{6} = \frac{62}{3}}$$

2)

$$\int_0^1 x\sqrt{1-x^2}\,dx = \int_0^1 x(1-x^2)^{1/2}dx$$

$$\frac{-1}{2}\int_0^1 -2x(1-x^2)^{1/2}dx \quad u = 1-x^2, du = -2x\,dx$$

$$\frac{-1}{2}\int_1^0 u^{1/2}du \qquad x=0, u=1 \quad x=1, u=0$$

$$\frac{1}{2}\left(\frac{2}{3}u^{3/2}\right)\Big|_0^1 = \frac{1}{3}$$

3)

$$\int_0^{\sqrt{5}} \frac{x}{\sqrt{x^2+4}}\,dx = \int_0^{\sqrt{5}} x(x^2+4)^{-1/2}dx$$

$$\frac{1}{2}\int_0^{\sqrt{5}} 2x(x^2+4)^{-1/2}dx \quad u = x^2+4, du = 2x\,dx$$

$$\frac{1}{2}\int_4^9 u^{-1/2}du \qquad x=0, u=4 \quad x=\sqrt{5}, u=9$$

$$\frac{1}{2}\left(2u^{1/2}\right)\Big|_4^9 = 1$$

4)

$$\int_0^{\pi/4} \sin 2x\,dx$$

$$\frac{1}{2}\int_0^{\pi/4} 2\sin 2x\,dx \qquad u = 2x, du = 2x\,dx$$

$$\frac{1}{2}\int_0^{\pi/2} \sin u\,du \qquad x=0, u=0 \quad x=\pi/4, u=\pi/2$$

$$\frac{1}{2}(-\cos u)\Big|_0^{\pi/2} = \frac{1}{2}$$

5)

$$\int_0^{\pi/2} \sin x\sqrt{\cos x}\,dx = \int_0^{\pi/2} \sin x(\cos x)^{1/2}\,dx$$

$$-\int_0^{\pi/2} -\sin x(\cos x)^{1/2}\,dx \qquad u = \cos x, du = -\sin x\,dx$$

$$-\int_0^{\pi/2} u^{1/2}\,du \qquad x=0, u=1 \quad x=\pi/2, u=0$$

$$\frac{2}{3}u^{3/2}\Big|_0^1 = \frac{2}{3}$$

6)

$$\int_0^{\pi/6} (1+\sin x\cos x)\,dx = \int_0^{\pi/6} 1\,dx + \int_0^{\pi/6} \sin x\cos x\,dx$$

$$-\int_0^{\pi/6} -\sin x(\cos x)^{1/2}\,dx \qquad u = \sin x, du = \cos x\,dx$$

$$\int_0^{\pi/6} 1\,dx + \int_0^{\pi/6} u\,du \qquad x=0, u=0 \quad x=\pi/6, u=1/2$$

$$x\Big|_0^{\pi/6} + \frac{u^2}{2}\Big|_0^{.5} = \frac{\pi}{6} + \frac{1}{8}$$

7)

$$\int_{-2}^{2} \left|1-x^2\right| dx$$

$$1-x^2 = 0 \Rightarrow x = \pm 1$$

$$2\left[\int_0^1 (1-x^2)\,dx + \int_1^2 (x^2-1)\,dx\right] = 4$$

$$2\left(\left[x-\frac{x^3}{3}\right]_0^1 + \left[\frac{x^3}{3}-x\right]_1^2\right)$$

$$2\left[1-\frac{1}{3}+\frac{8}{3}-2-\left(\frac{1}{3}-1\right)\right] = 4$$

Definite Integration with u-Substitution - Homework

Find the values of the following definite integrals. Verify using your calculator. Some will use *u*-substitution, others will not.

1. $\int_{-2}^{2}\left(x^3-1\right)dx$

$$\left[\frac{x^4}{4}-x\right]_{-2}^{2}$$
$$4-2-(4+2)=-4$$

2. $\int_{0}^{4}x\left(\sqrt{x}-1\right)dx$

$$\left[\frac{2x^{5/2}}{5}-\frac{x^2}{2}\right]_0^4$$
$$\frac{64}{5}-8-0=\frac{24}{5}$$

3. $\int_{0}^{\pi/3}\sin(2x)\,dx$

$$\frac{1}{2}\int_0^{\pi/3}2\sin 2x\,dx \qquad u=2x, du=2\,dx$$
$$\frac{1}{2}\int_0^{2\pi/3}\sin u\,du \qquad x=0, u=0 \quad x=\pi/3, u=2\pi/3$$
$$\frac{1}{2}(-\cos u)\Big|_0^{2\pi/3}=\frac{1}{4}+\frac{1}{2}=\frac{3}{4}$$

4. $\int_{0}^{\pi/12}(1-\cos 2x)\,dx$

$$x\Big|_0^{\pi/12}-\frac{1}{2}\int_0^{\pi/12}2\cos 2x\,dx \quad | \, u=2x, du=2x\,dx$$
$$\frac{\pi}{12}-\frac{1}{2}\int_0^{\pi/6}\cos u\,du \quad | \, x=0, u=0 \, | \, x=\pi/12, u=\pi/6$$
$$\frac{\pi}{12}-\frac{1}{2}(\sin u)\Big|_0^{\pi/6}=\frac{\pi}{12}-\frac{1}{4}$$

5. $\int_{0}^{1}2x\left(x^2+1\right)^2dx$

$$u=x^2+1, du=2x\,dx$$
$$\int_0^1 2x\left(x^2+1\right)^2\,dx=\int_0^2 u^2du$$
$$x=0, u=1 \, | \, x=1, u=2$$
$$\left(\frac{u^3}{3}\right)\Bigg|_1^2=\frac{8}{3}-\frac{1}{1}=\frac{7}{3}$$

6. $\int_{0}^{3}x\sqrt{9-x^2}\,dx$

$$u=9-x^2, du=-2x\,dx$$
$$\frac{-1}{2}\int_0^1-2x\left(9-x^2\right)^{1/2}\,dx$$
$$x=0, u=9 \, | \, x=3, u=0$$
$$\frac{-1}{2}\int_9^0 u^{1/2}du=\frac{1}{2}\left(\frac{2u^{3/2}}{3}\right)\Bigg|_0^9=9$$

7. $\int_{0}^{5}|x-4|\,dx$

$$\int_0^4(4-x)\,dx+\int_4^5(x-4)\,dx$$
$$\left[4x-\frac{x^2}{2}\right]_0^4+\left[\frac{x^2}{2}-4x\right]_4^5$$
$$16-8+\frac{25}{2}-20-(8-16)=\frac{17}{2}$$

8. $\int_{0}^{4}\left|x-\sqrt{x}\right|dx$

$$\int_0^1\left(\sqrt{x}-x\right)dx+\int_1^4\left(x-\sqrt{x}\right)dx$$
$$\left[\frac{2x^{3/2}}{3}-\frac{x^2}{2}\right]_0^1+\left[\frac{x^2}{2}-\frac{2x^{3/2}}{3}\right]_1^4$$
$$\frac{2}{3}-\frac{1}{2}+8-\frac{16}{3}-\left(\frac{1}{2}-\frac{2}{3}\right)=3$$

9. $\int_{2}^{3}\frac{x}{\left(x^2-3\right)^2}dx$

$$\int_2^3\frac{x}{\left(x^2-3\right)^2}dx=\frac{5}{12}$$
$$u=x^2-3, du=2x\,dx$$
$$\frac{1}{2}\int_2^3 2x\left(x^2-3\right)^{-2}\,dx$$
$$x=2, u=1 \, | \, x=3, u=6$$
$$\frac{1}{2}\int_1^6 u^{-2}du=\frac{1}{2}\left(\frac{-1}{u}\right)\Bigg|_1^6=\frac{-1}{12}+\frac{1}{2}=\frac{5}{12}$$

10. $\int_{0}^{4}\frac{dt}{\sqrt{2t+1}}$

$$u=2t+1, du=2\,dt$$
$$\frac{1}{2}\int_0^4 2(2t+1)^{-1/2}\,dt$$
$$t=0, u=1 \, | \, t=4, u=9$$
$$\frac{1}{2}\int_1^9 u^{-1/2}du=\left(u^{1/2}\right)\Big|_1^9=3-1-2$$

11. $\int_{0}^{\pi/2}\cos^3 t\sin t\,dt$

$$u=\cos t, du=-\sin t\,dt$$
$$-\int_0^{\pi/2}-\cos^3 t\sin t\,dt$$
$$t=0, u=1 \, | \, t=\pi/2, u=0$$
$$-\int_1^0 u^3du=\left(\frac{u^4}{4}\right)\Bigg|_0^1=\frac{1}{4}$$

12. $\int_{0}^{\sqrt{\pi/2}}t\sin\left(\pi-t^2\right)dt$

$$u=\pi-t^2, du=-2t\,dt$$
$$-\frac{1}{2}\int_0^{\sqrt{\pi/2}}2t\sin\left(\pi-t^2\right)dt$$
$$t=0, u=\pi \, | \, t=\sqrt{\pi/2}/2, u=\pi/2$$
$$-\frac{1}{2}\int_\pi^{\pi/2}\sin u\,du=\frac{1}{2}(\cos u)\Big|_\pi^{\pi/2}=\frac{1}{2}(0+1)=\frac{1}{2}$$

13. $\int_0^{\pi/4} \sqrt{\tan x}\,\sec^2 x\,dx$

14. $\int_0^{\pi/3} \cos x\sqrt{1-\cos^2 x}\,dx$

15. $\int_0^{1} x\sqrt{ax^2+b}\,dx$

13.
$u = \tan x, du = \sec^2 x\,dx$
$x = 0, u = 0 \mid x = \pi/4, u = 1$
$\int_0^1 u^{1/2}du = \left(\frac{2u^{3/2}}{3}\right)\Big|_0^1 = \frac{2}{3}$

14.
$\int_0^{\pi/3} \cos x \sin x\,dx$
$u = \sin x, du = \cos x\,dx$
$x = 0, u = 0 \mid x = \pi/3, u = \sqrt{3}/2$
$\int_0^{\sqrt{3}/2} u\,du = \left(\frac{u^2}{2}\right)\Big|_0^{\sqrt{3}/2} = \frac{3}{8}$

15.
$u = ax^2 + b, du = 2ax\,dx$
$\frac{1}{2a}\int_0^1 2ax\sqrt{ax^2+b}\,dx$
$x = 0, u = b \mid x = 1, u = a + b$
$\frac{1}{2a}\int_b^{a+b} u^{1/2}du = \left(\frac{1}{2a}\right)\left(\frac{2u^{3/2}}{3}\right)\Big|_b^{a+b} = \frac{\left(ax^2+b\right)^{3/2} - b^{3/2}}{3a}$

16. $\int_{\pi^2/4}^{\pi^2} \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

17. $\int_0^4 \left|9 - x^2\right| dx$

18. $\int_{-4}^{4} \frac{1}{x^2}\,dx$ (Be careful!)

16.
$u = \sqrt{x}, du = \frac{1}{2\sqrt{x}}dx$
$2\int_{\pi^2/4}^{\pi^2} \frac{\sin\sqrt{x}}{2\sqrt{x}}\,dx$
$x = \pi^2/4, u = \pi/2 \mid x = \pi^2, u = \pi$
$2\int_{\pi/2}^{\pi} \sin u\,du = \left(-2\cos u\right)\Big|_{\pi/2}^{\pi} = 2$

17.
$9 - x^2 = 0 \Rightarrow x = \pm 3$
$\int_0^3 \left(9 - x^2\right)dx + \int_3^4 \left(x^2 - 9\right)dx$
$\left[9x - \frac{x^3}{3}\right]_0^3 + \left[\frac{x^3}{3} - 9x\right]_3^4$
$27 - 9 + \frac{64}{3} - 36 - \left(9 - 27\right) = \frac{64}{3}$

18.
Since $\frac{1}{x^2}$ is not continuous at $x = 0$, the definite integral which represents area under the curve makes no sense.

If $\int_0^2 f(x)\,dx = \frac{11}{3}$ and $\int_0^6 f(x)\,dx = 15$, $f(x)$ is an even function (symmetric to the y - axis), find the following:

19. $\int_{-2}^{0} f(x)\,dx = \frac{11}{3}$ 20. $\int_{-2}^{2} f(x)\,dx = \frac{22}{3}$ 21. $\int_0^2 -f(x)\,dx = \frac{-11}{3}$ 22. $\int_{-2}^{0} 3f(x)\,dx = 11$ 23. $\int_0^2 f(3x)\,dx = 5$

If $\int_0^2 f(x)\,dx = \frac{11}{3}$ and $\int_0^6 f(x)\,dx = 15$, $f(x)$ is an odd function (symmetric to the origin), find the following:

24. $\int_{-2}^{0} f(x)\,dx = \frac{-11}{3}$ 25. $\int_{-2}^{2} f(x)\,dx = 0$ 26. $\int_0^2 -f(x)\,dx = \frac{-11}{3}$ 27. $\int_{-2}^{0} 3f(x)\,dx = -11$ 28. $\int_{-2}^{2} f(3x)\,dx = 0$

23.
$u = 3x \quad du = 3dx \quad \frac{1}{3}\int_0^2 3f(3x)\,dx$
$x = 0, u = 0 \mid x = 2, u = 6$
$\frac{1}{3}\int_0^6 f(u)\,du = \frac{1}{3}(15) = 5$

28.
$u = 3x \quad du = 3dx \quad \frac{1}{3}\int_{-2}^{2} 3f(3x)\,dx$
$x = -2, u = -6 \mid x = 2, u = 6$
$\frac{1}{3}\int_{-6}^{6} f(u)\,du = \frac{1}{3}(0) = 0$

Straight Line Motion - Revisited - Classwork

When we looked at straight line motion in our derivative section, we generated a relationship between position, velocity, and acceleration. Given position function $s(t)$, the velocity function $v(t) = s'(t)$ and the acceleration function $a(t) = v'(t) = s''(t)$. We can now move in the opposite direction as well. If you have an object traveling in a straight line, its velocity can be written as $v(t) = \int a(t)\,dt + C$..

Examples) Find $v(t)$ given $a(t)$ and $v(0)$

1) $a(t) = 4t - 6$ and $v(0) = 3$

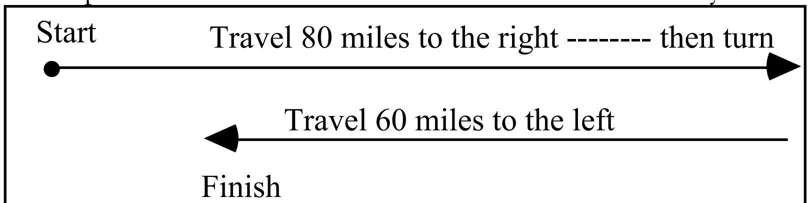
$$v(t) = 2t^2 - 6t + C$$
$$v(0) = C = 3$$
$$v(t) = 2t^2 - 6t + 3$$

2) $a(t) = \sin t + 2t$ and $v(0) = 5$

$$v(t) = -\cos t + t^2 + C$$
$$v(0) = -1 + C = 5 \Rightarrow C = 6$$
$$v(t) = -\cos t + t^2 + 6$$

We now define two terms: distance: how far you travel between time $t = a$ and time $t = b$.

displacement: the difference in distance from where you start and where you stop.



Displacement = 20 miles (80 - 60)
Distance = 140 miles

Can displacement equal distance? How? When motion is always forward

So if we are given a velocity function which is always positive, we know that distance = displacement. But if the velocity function is not always positive, we must find when the time interval when the velocity is negative (moving backwards) so we can find the distance traveled in that interval and subtract it (which will be subtracting a negative number or adding a positive one). This can be summarized as:

Given a particle moving on a straight line with velocity $v(t)$ between time $t = a$ and time $t = b$. Then

$$\text{Displacement} = \int_a^b v(t)\,dt \qquad \text{Distance} = \int_a^b |v(t)|\,dt$$

Example 3) A particle is moving along a straight line with velocity $v(t) = t^2 - 7t + 10$ (ft/sec) Find the displacement and distance traveled on the time interval $[1,7]$

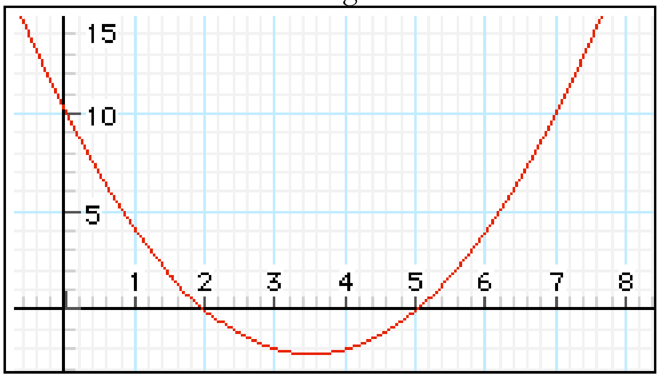
Displacement (easy)

$$\text{Disp} = \int_1^7 (t^2 - 7t + 10)\,dt = 6$$

Distance (more difficult)

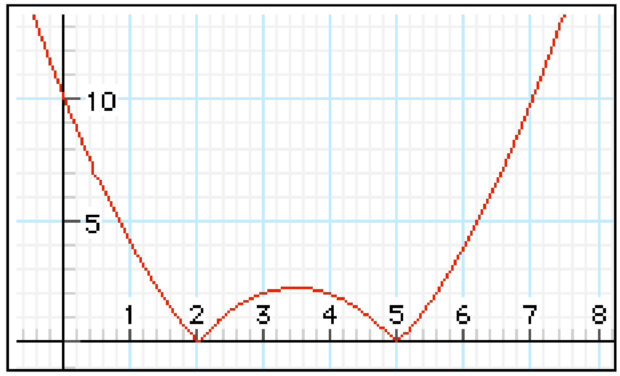
$$\text{Distance} = \int_1^2 |t^2 - 7t + 10|\,dt$$
$$\text{Distance} = \int_1^2 (t^2 - 7t + 10)\,dt - \int_2^5 (t^2 - 7t + 10)\,dt + \int_5^7 (t^2 - 7t + 10)\,dt = 15$$

This can be confirmed using the calculator.



This is the graph of $v = t^2 - 7t + 10$, notice that v is negative (going backward) between $t = 2$ and $t = 5$. When you take the integral $\int_1^7 (t^2 - 7t + 10)\,dt$, this will be taken into account.

If you are allowed to use your calculator, you use FnInt(X2-7X+10,X,1,7)



This is the graph of $v = |t^2 - 7t + 10|$. Note that between $t = 2$ and $t = 5$, the function is now above the axis and distance is positive. So the integral is

$$\int_1^7 |t^2 - 7t + 10|\,dt =$$

$$\int_1^2 (t^2 - 7t + 10)\,dt - \int_2^5 (t^2 - 7t + 10)\,dt + \int_5^7 (t^2 - 7t + 10)\,dt$$

If you are allowed to use your calculator, you use FnInt(abs(X2-7X+10),X,1,7)

Example 4) Given an object moving in a straight line with $a(t) = \sqrt{t}\ \frac{\text{ft}}{\text{sec}^2}, v(0) = -18,\ t = 0$ to $t = 16$, find $v(t)$ and the displacement and distance of the object.

$$v(t) = \frac{2}{3}t^{3/2} + C \Rightarrow v(0) = C = -18$$

$$v(t) = \frac{2}{3}t^{3/2} - 18$$

$$\text{Disp} = \int_0^{16} \left(\frac{2}{3}t^{3/2} - 18\right) dt = -14.933 \text{ ft}$$

$$\text{Dist} = \int_0^{16} \left|\frac{2}{3}t^{3/2} - 18\right| dt$$

$$\text{Dist} = -\int_0^{9} \left(\frac{2}{3}t^{3/2} - 18\right) dt + \int_9^{16} \left(\frac{2}{3}t^{3/2} - 18\right) dt = 179.467 \text{ ft}$$

Example 5) A subway train accelerates as it leaves one station, then decelerates as it comes into the next station. following chart measures the velocity v given in miles per hour .

time t (sec)	0	5	10	15	20	25	30	35	40	45	50	55	60
velocity	0	15	33	42	44	44	44	44	43	38	24	8	0
Distance	**0**	**.010**	**.033**	**.052**	**.060**	**.061**	**.061**	**.061**	**.060**	**.056.**	**.043**	**.022**	**.006**

a) find the distance the train travels every 5 second interval.

Use the trapezoid rule for each entry. For instance, the first entry is

$$\left(\frac{1}{2}\right)5 \text{ sec}\frac{1 \text{ hr}}{3600 \text{ sec}}\left(15\frac{\text{miles}}{\text{hr}} - 0\right) = .010$$

b) find the total distance between subway stops.

Either add up all the results or use :

$$\left(\frac{1}{2}\right)5 \text{ sec}\frac{1 \text{ hr}}{3600 \text{ sec}}\left(0 + 2(15) + 2(33) + \ldots + 2(24) + 2(8) + 0\right) = .526 \text{ miles}$$

Straight Line Motion - Revisited - Homework

Given the velocity of an object in ft/sec, find the displacement and distance traveled in the given time interval.

1. $v(t) = 12 - 3t \quad [0, 5]$

$$\text{Disp} = \int_0^5 (12 - 3t)\, dt = 22.5 \text{ ft}$$

$$\text{Dist} = \int_0^5 |12 - 3t|\, dt$$

$$\text{Dist} = \int_0^4 (12 - 3t)\, dt - \int_4^5 (12 - 3t)\, dt = 25.5 \text{ ft}$$

2. $v(t) = t^2 - 10t + 16 \quad [0, 6]$

$$\text{Disp} = \int_0^6 (t^2 - 10t + 16)\, dt = -12 \text{ ft}$$

$$\text{Dist} = \int_0^6 |t^2 - 10t + 16|\, dt$$

$$\text{Dist} = \int_0^2 (t^2 - 10t + 16)\, dt - \int_2^6 (t^2 - 10t + 16)\, dt = 41.333 \text{ ft}$$

Given the acceleration of an object in ft/sec^2 and its initial velocity, find the displacement and distance traveled in the given time interval.

3. $a(t) = 4t, v(0) = -8 \quad [0,3]$

$$v(t) = 2t^2 - 8 \qquad \text{Disp} = \int_0^3 (2t^2 - 8)\, dt = -6 \text{ ft}$$

$$\text{Dist} = \int_0^3 |2t^2 - 8|\, dt$$

$$\text{Dist} = -\int_0^2 (2t^2 - 8)\, dt + \int_2^3 (2t^2 - 8)\, dt = 15.333 \text{ ft}$$

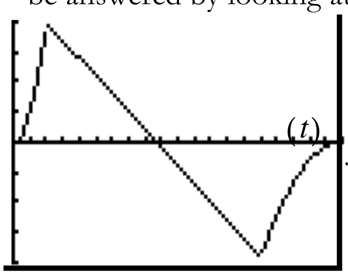
4. $a(t) = 6\sin t, v(0) = -9 \quad [0,\pi]$

$$v(t) = -6\cos t + C, C = -3, v(t) = -6\cos t - 3$$

$$\text{Disp} = \int_0^\pi (-6\cos t - 3)\, dt = -9.425 \text{ ft}$$

$$\text{Dist} = \int_0^\pi |-6\cos t - 3|\, dt = 13.534 \text{ ft}$$

5) When a model rocket is launched, the propellant burns for a few seconds, accelerating the rocket upward. After burnout the rocket coasts upward for awhile and then begins to fall. A small explosive charge pops out a parachute while the rocket is on its way down. The parachute slows the rocket to keep it from smashing when it lands. The velocity function is given below. Using your calculator, sketch it. Questions a through g should be answered by looking at your graph. Question h) and i) need graphing calculators.



$$f(t) = \begin{cases} 50t^2, t \le 2 \\ 260 - 30t, 2 < t \le 15 \\ -7.6(t-20)^2, t > 15 \end{cases}$$

a) How many seconds did the rocket burn? **2 sec**

b) How fast was the rocket moving when the engine stopped? **200 ft/sec**

c) When did the rocket reach hit its highest point? **$8\frac{2}{3}$ sec**

d) When did the parachute pop out? **15 sec**

e) How long did the rocket fall before the parachute opened? **$6\frac{1}{3}$ sec**

f) How fast was the rocket falling when the parachute opened? **190 ft/sec**

g) When was the rocket's acceleration the greatest? **2 sec**

h) Assuming the rocket does land, how far does it travel?

$$2\left[\int_0^2 50t^2 dt + \int_2^{8.667} (260 - 30t)\, dt\right] = \boxed{1600 \text{ ft}}$$

i) At what time after launch did the rocket land?

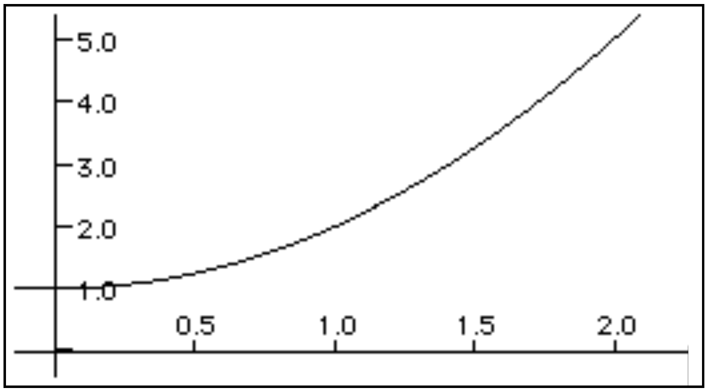
$$\int_{8.667}^{15} (260 - 30t)\, dt + \int_{15}^{k} -7.6(t-20)^2\, dt = -800$$

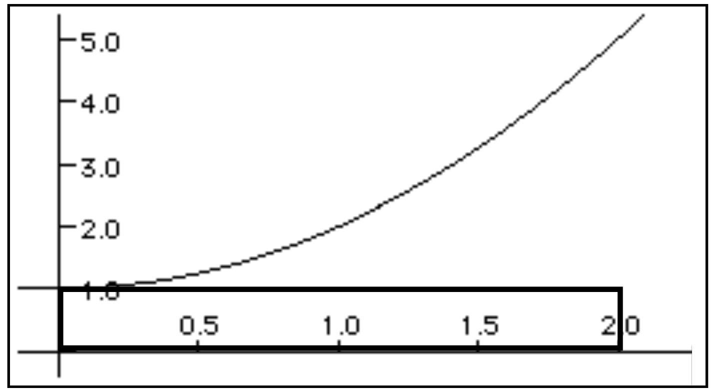
$$-2.533(t-20)^3\Big|_{15}^{k} = -198.333 \Rightarrow (k-20)^3 - (-5)^3 = 78.289$$

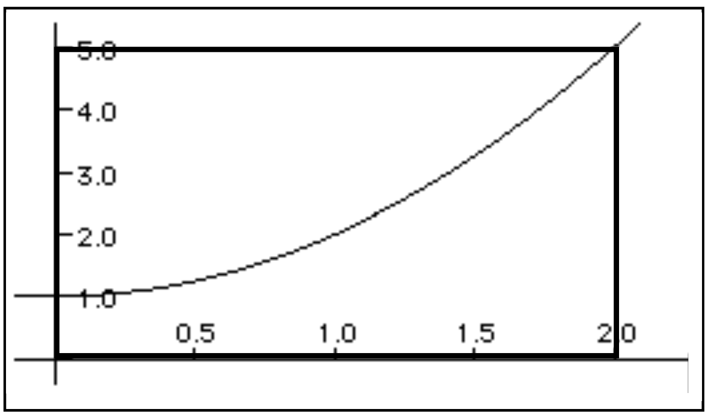
$$\boxed{k = 16.398 \text{ sec}}$$

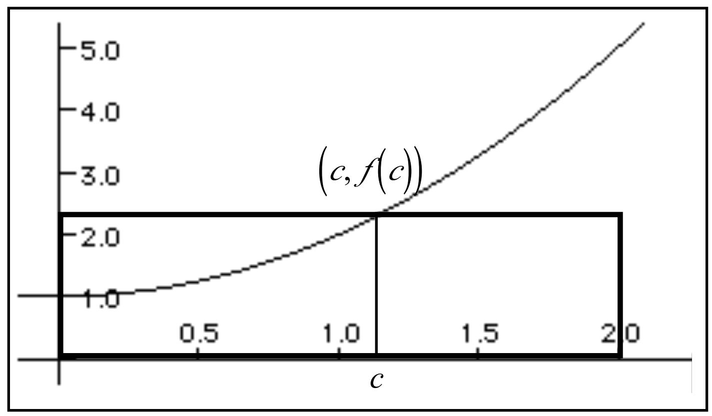
Average Value of a Function & 2nd Fund. Thm. - Classwork

On the graph below left, $f(x)$, we know that the area under the curve between 0 and 2 is $\int_0^2 f(x)\,dx$. The question is, is there a rectangle whose base is 2 which is exactly equal to this integral? Clearly, the area under the rectangle below right is smaller than the area under $f(x)$.









Clearly, the area under the rectangle is greater than the area under the curve $f(x)$.

The area under the curve in this picture is now fairly close to the area under the rectangle.

This can be summarized by the **Mean Value Theorem for Integrals** (not to be confused with the MVT for derivatives).

If f is continuous on the closed interval $[a,b]$, then there exists a number c on $[a,b]$ such that $\int_a^b f(x)\,dx = f(c)(b-a)$. What this says is that the area under the curve between a and b can be expressed as the base of the rectangle $(b-a)$ times the height of the rectangle at some point $(c, f(c))$.

Since $\int_a^b f(x)\,dx = f(c)(b-a)$, we can write $\dfrac{\int_a^b f(x)\,dx}{b-a} = f(c)$. This $\dfrac{\int_a^b f(x)\,dx}{b-a}$ is called the average value of a function. It represents the height of the rectangle guaranteed by the MVT for integrals.

Example 1) Given $f(x) = x^2 + 1$ on the interval [0, 2] (the problem on the previous page), find
a) the average value of the function b) the value of *c* guaranteed by the MVT for integrals.

$$f_{ave} = \frac{\int_0^2 (x^2+1)\,dx}{2-0} = \frac{7}{3} \qquad c^2 + 1 = \frac{7}{3} \Rightarrow c = \frac{2}{\sqrt{3}}$$

Example 2) Given $f(x) = \sin x$ on the interval [0, π], find
a) the average value of the function b) the value of *c* guaranteed by the MVT for integrals.

$$f_{ave} = \frac{\int_0^\pi (\sin x)\,dx}{\pi - 0} = \frac{2}{\pi} \qquad \sin c = \frac{2}{\pi} \Rightarrow c = .690$$

Suppose you were asked to find $\int_1^x (t^2 - 4t + 1)\,dt$. Use the Fundamental Theorem of Calculus to do so.

$$\left.\frac{t^3}{3} - 2t^2 + t\right|_1^x = \frac{x^3}{3} - 2x^2 + x - \left(\frac{1}{3} - 2 + 1\right)$$

Now, take the derivative of your answer. $\int_1^x (t^2 - 4t + 1)\,dt$. What is the basic result? Integral and derivatives are inverses and cancel out.

The Second Fundamental Theorem of Calculus.

If f is continuous on an open interval *I* containing *a*, then, for every *x* in the interval,

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x) \text{ and } \frac{d}{dx}\left[\int_a^u f(t)\,dt\right] = f(u)\,\frac{du}{dx}$$

What this says that the constant on the bottom doesn't matter when we take the derivative of an integral. The derivative of the integral is the original integrand (but with the variable changed).

Example 3) Find the following.

a. $\frac{d}{dx}\left[\int_1^x \sqrt{t^2-1}\right] dt$

$\sqrt{x^2-1}$

b. $\frac{d}{dx}\left[\int_x^3 t\sin t\right] dt$

$-x\sin x$

c. $\frac{d}{dx}\left[\int_{\pi/2}^{x^2} \cos t\right] dt$

$2x\cos x^2$

Average Value of a Function & 2nd Fund. Thm. - Homework

1. Find the average value of the function f on the given interval. Do not use calculators.

a)
$$f(x) = x^2 - 2x \quad [0,3]$$
$$f_{ave} = \frac{\int_0^3 (x^2 - 2x)\, dx}{3-0} = 0$$

b)
$$f(x) = \cos x \quad \left[0, \frac{\pi}{2}\right]$$
$$f_{ave} = \frac{\int_0^{\pi/2} \cos x\, dx}{\pi/2 - 0} = \frac{2}{\pi}$$

c)
$$f(x) = \sqrt{x} \quad [1,9]$$
$$f_{ave} = \frac{\int_1^9 \sqrt{x}\, dx}{9-1} = \frac{13}{6}$$

d)
$$f(x) = \frac{1}{x^2} \quad [1,5]$$
$$f_{ave} = \frac{\int_1^5 \frac{1}{x^2}\, dx}{5-1} = \frac{1}{5}$$

2. Find the value of x guaranteed by the Mean Value Theorem for Integrals. No calculators on a) and b).

a)
$$f(x) = 4 - x^2 \quad [0,2]$$
$$f_{ave} = \frac{\int_0^2 (4 - x^2)\, dx}{2-0} = \frac{8}{3}$$
$$4 - x^2 = \frac{8}{3} \Rightarrow x = \frac{2}{\sqrt{3}}$$

b)
$$f(x) = 4x - x^2 \quad [0,3]$$
$$f_{ave} = \frac{\int_0^3 (4x - x^2)\, dx}{3-0} = 3$$
$$4x - x^2 = 3 \Rightarrow x = 1$$

c)
$$f(x) = x^3 - x + 1 \quad [0,2]$$
$$f_{ave} = \frac{\int_0^2 (x^3 - x + 1)\, dx}{2-0} = 2$$
$$x^3 - x + 1 = 2 \Rightarrow x = 1.325$$

d)
$$f(x) = x \sin x^2 \quad [0, \sqrt{\pi}]$$
$$f_{ave} = \frac{\int_0^{\sqrt{\pi}} (x \sin x^2)\, dx}{\sqrt{\pi} - 0} = \frac{1}{\sqrt{\pi}}$$
$$x \sin x^2 = \frac{1}{\sqrt{\pi}} \Rightarrow x = .851,\ 1.673$$

3. Find the numbers b such that the average value of $f(x) = 2 + 7x - x^3$ on the interval $[0, b]$ is equal to 3.

$$f_{ave} = \frac{\int_0^b (2 + 7x - x^3)\, dx}{b-0} = 3$$
$$\frac{2b + \frac{7b^2}{2} - \frac{b^4}{4}}{b} = 3 \Rightarrow b = -3.877, b = .287, b = 3.590$$

4. In a certain city, the temperature (in °F) t hours after 9 A.M is approximated by the function $T(t) = 50 + 14\sin\frac{\pi t}{12}$. Find the average temperature during the time period 9 AM to 9 PM.

$$f_{ave} = \frac{\int_0^{12}\left(50 + 14\sin\frac{\pi t}{12}\right)dx}{12-0} = 58.913^o$$

5. Show that the Second Fundamental Theorem of Calculus holds for $F(x) = \int_1^x \frac{2}{t^2}\,dt$. Take the integral, then take the derivative.

$$\frac{d}{dx}\int_1^x \frac{2}{t^2}\,dt = \frac{d}{dx}\left[\frac{-2}{t}\right]_1^x$$

$$\frac{d}{dx}\left[\frac{-2}{x} + 2\right]$$

$$\frac{2}{x^2}$$

6. Use the Second Fundamental Theorem of Calculus to find the derivatives of the following functions.

a) $f(x) = \int_1^x \left(t^2+1\right)^{20} dt$

$f'(x) = \left(x^2+1\right)^{20}$

b) $g(x) = \int_{-1}^x \sqrt{t^3+1}\,dt$

$g'(x) = \sqrt{x^3+1}$

c) $g(x) = \int_\pi^x \frac{1}{1+t^4}\,dt$

$g'(x) = \frac{1}{1+x^4}$

d) $f(x) = \int_4^{x^2} \cos\left(t^2\right) dt$

$f'(x) = 2x\cos\left(x^4\right)$

7. Find the interval on which the curve $y = \int_0^x \left(t^3+t^2+1\right) dt$ is concave up. Justify your answer.

$$y = \int_0^x \left(t^3+t^2+1\right) dt$$

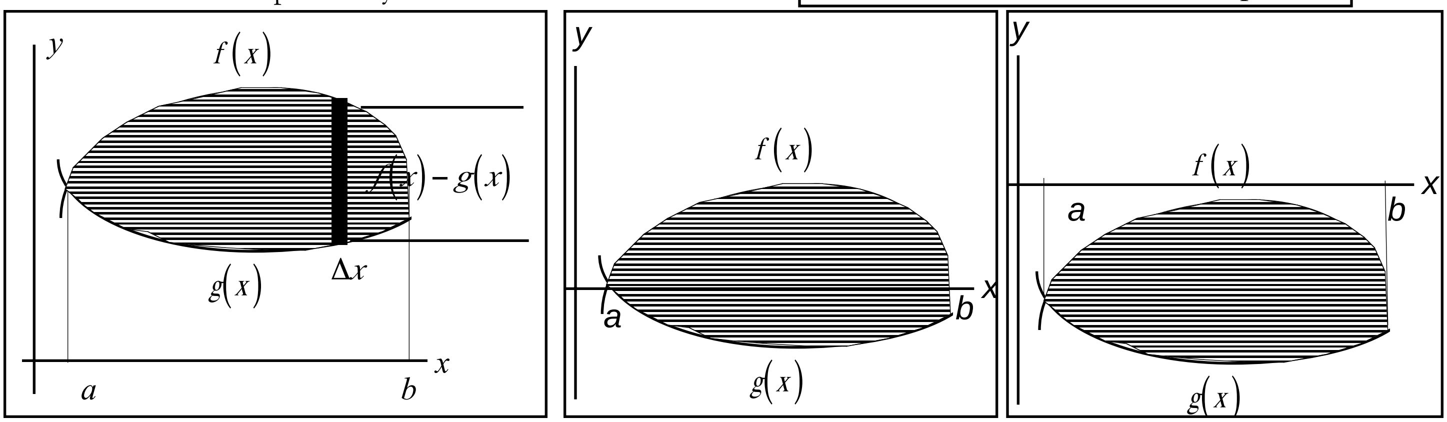
$$y' = x^3 + x^2 + 1$$

$$y'' = 3x^2 + 2x = 0 \Rightarrow \left(-\infty, -\frac{2}{3}\right) \cup (0, \infty)$$

<u>Area of a Region Between Two Curves - Classwork</u>

Consider two functions f and g that are continuous on $[a,b]$. If the graphs of both functions are above the *x*-axis and $f(x) \geq g(x)$, we can find the area between the two graphs as the area of the region under the graph of g subtracted from the area of f. Picture below left. The area of a represented rectangle is width times height. The base is Δx and the height of the rectangle is $f(x)-g(x)$. So the area of the rectangle is $\left(f(x)-g(x)\right)\Delta x$. As Δx goes to zero, we have a Riemann sum - $\lim_{n\to\infty}\left[\sum_{i=1}^{n}\left(f(x_i)-g(x_i)\right)\Delta x\right]$ which we know to be $A=\int_a^b\left[f(x)-g(x)\right]dx$.

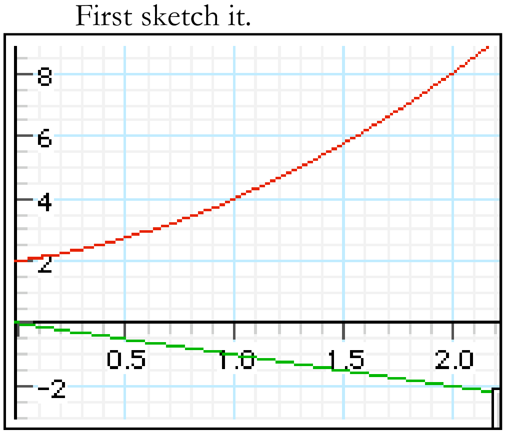
So we say that the area of the shaded region is $A=\int_a^b\left[f(x)-g(x)\right]dx$. Now suppose the curves are not both above the *x*-axis? Explain why that doesn't make a difference. the area is the same no matter the position.



$A=\int_a^b\left[f(x)-g(x)\right]dx$ $\qquad$ $A=\int_a^b\left[f(x)-g(x)\right]dx$ $\qquad$ $A=\int_a^b\left[f(x)-g(x)\right]dx$

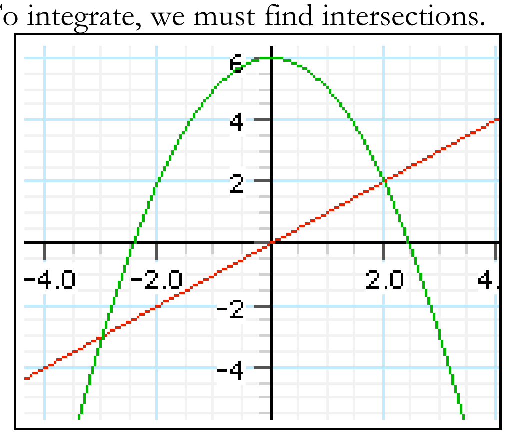
As we go through problems in this section, we will use extensively the calculator's ability to find numerical integrals. We will not go through the FTC - integrating and plugging in the top and bottom limits. On the A.P. test, you are allowed to use the calculator's ability to numerically integrate as long as you show the proper setup.

Example 1) Find the region bounded by the graphs of $y=x^2+x+2, y=-x, x=0$ and $x=1$. First sketch it.



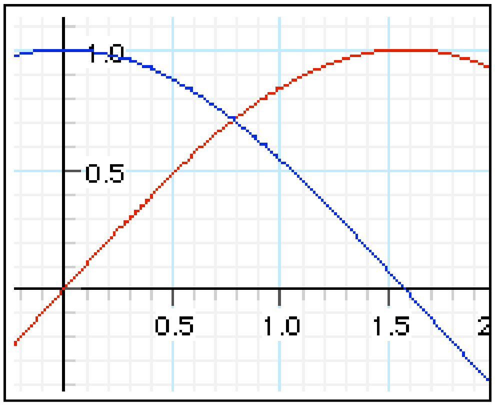
The integral is $\int_0^1\left[\left(x^2+x+2\right)-(-x)\right]dx=\frac{10}{3}$

Example 2) Find the region bounded by the graphs $y=6-x^2$ and $y=x$. Sketch it. To integrate, we must find intersections.



The integral is $\int_{-3}^{2}\left[6-x^2-x\right]dx=\frac{125}{6}$

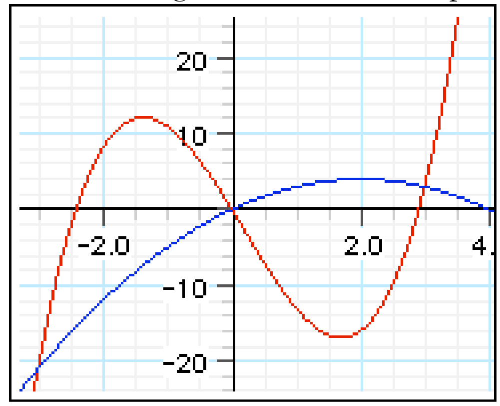
Example 3) Find the region bounded by the graphs $x = 0$, and the first intersection of $y = \sin x$ and $y = \cos x$.



Your first job is to find the intersection of these two curves. Do it algebraically.

The integral is $\displaystyle\int_0^{\pi/4}\left[\cos x - \sin x\right]dx = \sqrt{2} - 1$

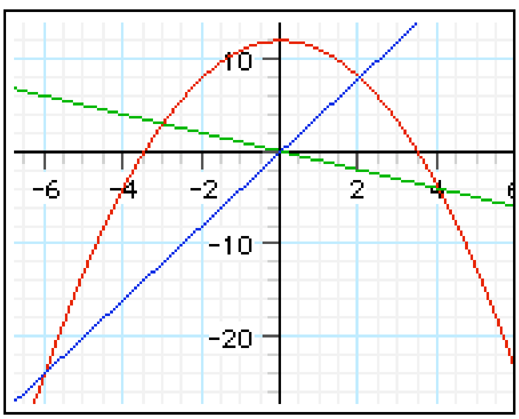
Example 4) Find the region bounded by $y = 2x^3 - x^2 - 14x$ and $y = 4x - x^2$. Sketch it noting which curve is on top.



Find the intesections.

The integral is $\displaystyle\int_{-3}^{0}\left[2x^3 - x^2 - 14x - \left(4x - x^2\right)\right]dx + \int_0^3\left[4x - x^2 - \left(2x^3 - x^2 - 14x\right)\right]dx = 81$

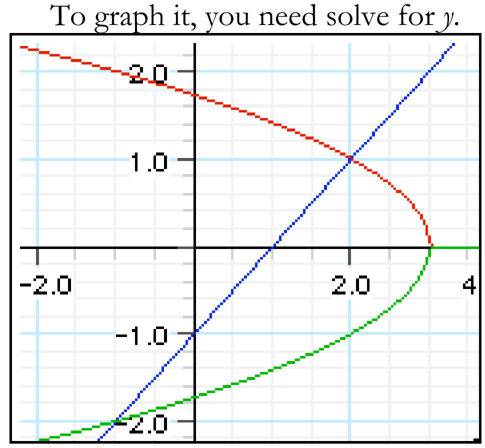
Example 5) Find the region bounded by the graphs $y = 12 - x^2$, $y = -x$, and $y = 4x$.



Shade the region you are trying to find.

$\displaystyle\int_{-6}^{0}\left[12 - x^2 - 4x\right]dx + \int_0^4\left[12 - x^2 + x\right]dx = 106.667$

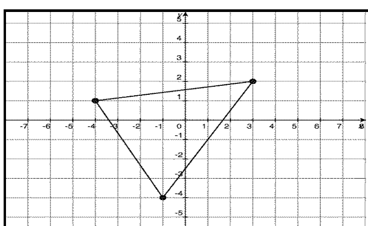
Example 6) Find the region bounded by $x = 3 - y^2$ and $x = y + 1$. To graph it, you need solve for *y*.



Rather than integrating using *x*, use *y* and go right to left instead of top to bottom

$\displaystyle\int_{-2}^{1}\left[3 - y^2 - (y + 1)\right]dy = 4.5$

Example 7. On the graph below, choose 3 points at random, draw the triangle, and find its area.



Pts: $(-4,1),(3,2),(-1.-4)$

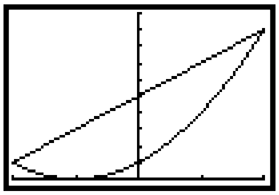
Lines: $y = \dfrac{x+11}{7}$, $y = \dfrac{-5x-17}{3}$, $y = \dfrac{3x-5}{2}$

$$A = \int_{-4}^{-1}\left(\frac{x+11}{7} - \frac{-5x-17}{3}\right) + \int_{-1}^{3}\left(\frac{x+11}{7} - \frac{3x-5}{2}\right) = \frac{57}{7} + \frac{76}{7} = \frac{190}{7}$$

Area of a Region Between Two Curves - Homework

For each problem, sketch the region bounded by the graphs of the functions and find the region of the area.

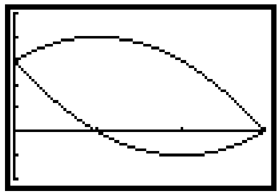
1. $y = x^2 + 2x + 1, y = 2x + 5$



$$\int_{-2}^{2}\left[2x + 5 - \left(x^2 + 2x + 1\right)\right]dx$$

Area $= 10.667$

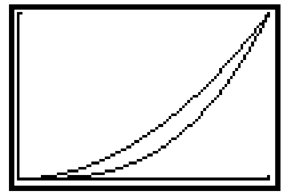
2. $y = x^2 - 4x + 3,$ $y = -x^2 + 2x + 3$



$$\int_{0}^{3}\left[-x^2 + 2x + 3 - \left(x^2 - 4x + 3\right)\right]dx$$

Area $= 9$

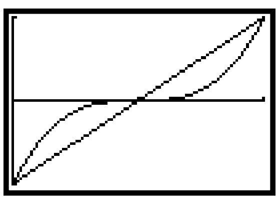
3. $y = x^2$ $y = x^3$



$$\int_{0}^{1}\left[x^2 - x^3\right]dx$$

Area $= \frac{1}{12}$

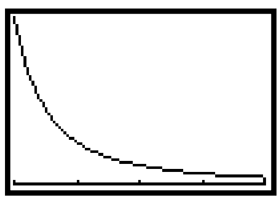
4. $y = \left(x-1\right)^3, y = x - 1$



$$2\int_{0}^{1}\left[\left(x-1\right)^3 - \left(x-1\right)\right]dx$$

Area $= \frac{1}{2}$

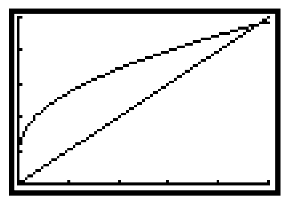
5. $y = \frac{1}{x^2}, y = 0, x = 1, x = 5$



$$\int_{1}^{5}\left[\frac{1}{x^2}\right]dx$$

Area $= \frac{4}{5}$

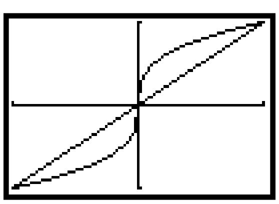
6. $y = \sqrt{3x} + 1, y = x, x = 0$



$$\int_{0}^{4.791}\left[\sqrt{3x} + 1 - x\right]dx$$

Area $= 5.423$

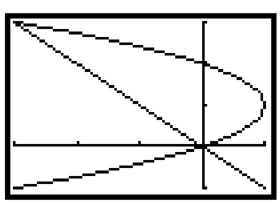
7. $y = \sqrt[3]{x}, y = x$



$$2\int_{0}^{1}\left[\sqrt[3]{x} - x\right]dx$$

Area $= \frac{1}{2}$

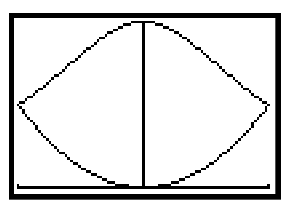
8. $x = 2y - y^2, x = -y$



$$\int_{0}^{3}\left[2y - y^2 + y\right]dy$$

Area $= \frac{9}{2}$

9. $y = \frac{1}{1 + x^2}, y = \frac{1}{2}x^2$



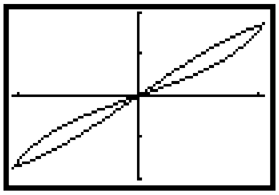
$$2\int_{0}^{1}\left[\frac{1}{1+x^2} - \frac{1}{2}x^2\right]dx$$

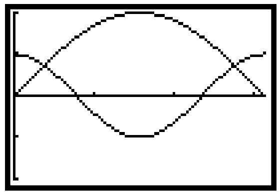
Area $= 1.237$

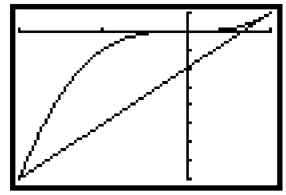
10. $y = 2\sin x, y = \tan x$ $-\frac{\pi}{3} \le x \le \frac{\pi}{3}$

11. $y = 2\sin x, y = \cos 2x$ $0 \le x \le \pi$

12. $y = x^3$ and the tangent line to y at $(1,1)$







$$2\int_0^{\pi/3}[\sin x - \tan x]\,dx$$
Area $= 0.614$

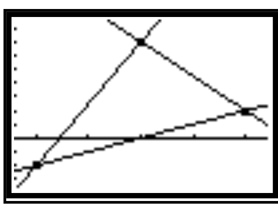
$$2\int_0^{.375}[\cos 2x - 2\sin x]\,dx + \int_{375}^{2.767}[2\sin x - \cos 2x]\,dx$$
Area $= 4.807$

$$\int_{-2}^{1}\left[x^3 - (3x-2)\right]dx$$
Area $= 6.750$

13. Find the value of b if the vertical line $x = b$ divides the region between $y = 16 - 2x$ and the x and y-axis into 2 equal areas.

$$\int_0^b (16-2x)\,dx = \int_b^8 (16-2x)\,dx$$
$$16b - b^2 = 128 - 64 - \left(16b - b^2\right)$$
$$0 = 2b^2 - 32b + 64$$
$$0 = b^2 - 16b + 32 \Rightarrow b = 2.343$$

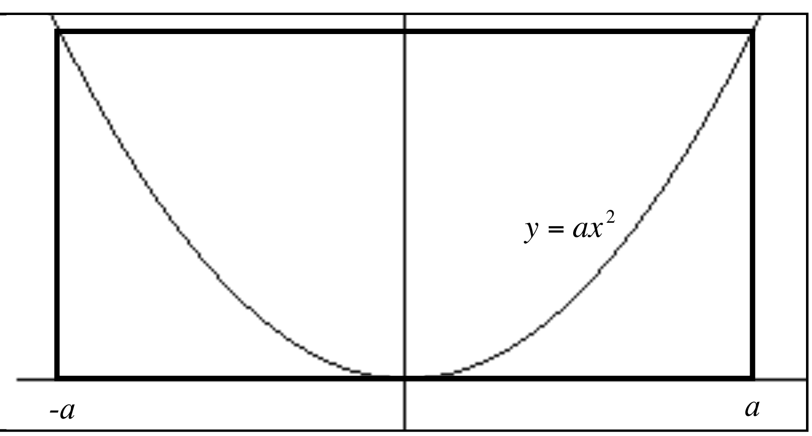
14. Use integration to to find the area of the triangle having the vertices (3, -2), (5, 7), and (7, 2).



$$\int_3^5\left[4.5x - \frac{31}{2} - (x-5)\right]dx + \int_5^7\left[-2.5x + \frac{39}{2} - (x-5)\right]dx = 14$$

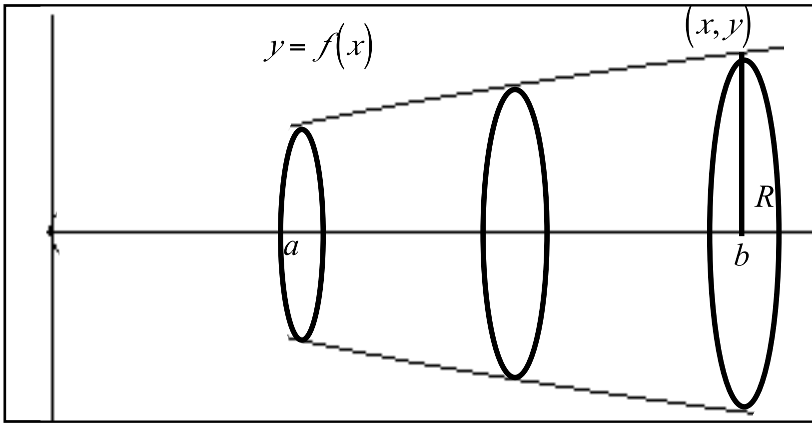
15. Show that the area under the function $y = ax^2$ is $\frac{1}{3}$ of the area of the circumscribed rectangle.

$$A = \int_0^a ax^2dx = \frac{2}{3}\left[ax^3\right]_0^a = \frac{2}{3}a^4$$
$$A_{\text{rect}} = 2a\left(a^3\right) = 2a^4$$

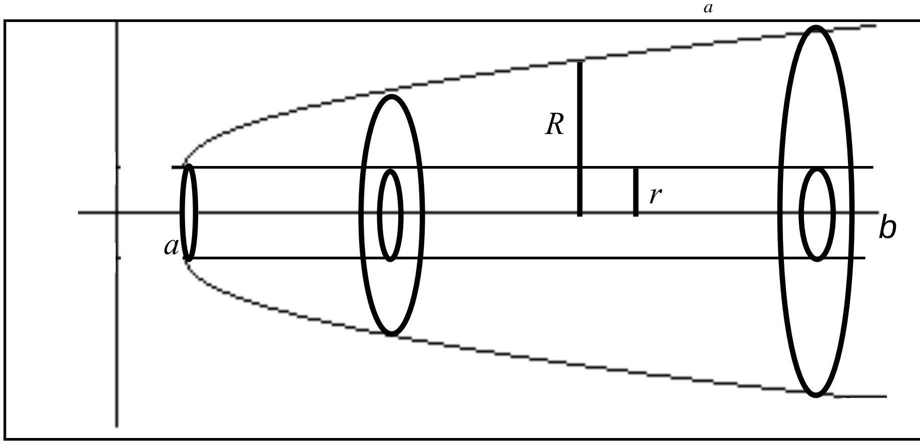


Volume by Disk/Washers - Classwork

1st problem: You are given the function $y = f(x)$ and you are looking at between $x = a$ and $x = b$. You are then going to rotate the function about the *x*-axis. The 3-D object you get appears like the one below. We wish to find the volume of the object. If you are to look at a cross section by slicing it perpendicular to the *x*-axis, we get a circle. The formal name for this circle is **a disk**. The area of this disk is πR^2 where R is the radius of the disk. In reality though, this disk is a 3-D shape with a width of Δx. So the volume of a representative disk is $\pi R^2 \cdot \Delta x$. If we take thinner and thinner disks, Δx gets close to zero. So the volume of the solid is $\lim_{n\to\infty}\sum_{i=1}^{n} \pi R_i^2 \cdot \Delta x$ which we know to be a Riemann sum. Our job is to find R. The way we will do this is to draw the Radius and label the endpoint as (x, y). So we end up with the disk formula: $V = \pi \int_a^b R^2 dx$



2nd problem: The problem remains the same except the region between two curves are rotated about the *x*-axis. Instead of creating a disk, we create a washer (2 concentric circles with "air" in between) Using the same argument, the area of the washer is $\pi\left(R^2 - r^2\right)$ where R is the outside radius and r is the inside radius. So, we end up with $V = \pi \int_a^b \left(R^2 - r^2\right) dx$

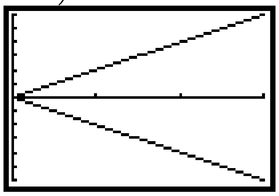


Generalizations:

1. Draw the function (s). **2**. Determine the region to rotate. **3a**. If it is rotated about the *x*-axis or any horizontal line, it is a *dx* problem and everything must be in terms of *x*. **3b** If it is rotated about the *y*-axis or any vertical line, it is a *dy* problem and everything must be in terms of *y*. **4a** If it is a disk problem, you need to find *R*. **4b** . If it is washer problem, you need to find *R* and *r*. **5**. When in doubt, label the point on the disk/washer as (x, y). *y* is a vertical distance and *x* is a horizontal distance.

Example 1) Find the volume if the region enclosing $y = 2x, y = 0, x = 3$ is rotated about the

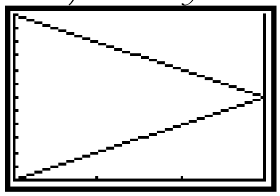
a) x-axis



$R = 2x$

$V = \pi\int_0^3 4x^2 dx = 36\pi$

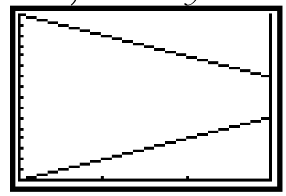
b) the line $y = 6$



$R = 6, r = 6 - 2x$

$V = \pi\int_0^3 \left[36 - (6 - 2x)^2\right] dx = 72\pi$

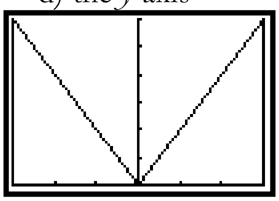
c) the line $y = 8$



$R = 8, r = 8 - 2x$

$V = \pi\int_0^3 \left[64 - (8 - 2x)^2\right] dx = 108\pi$

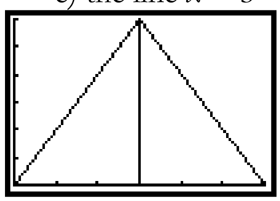
d) the y-axis



$R = 3, r = \dfrac{y}{2}$

$V = \pi\int_0^6 \left[9 - \dfrac{y^2}{4}\right] dy = 36\pi$

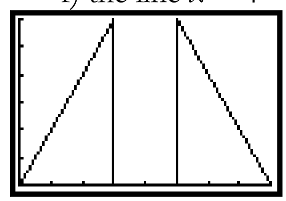
e) the line $x = 3$



$R = 3 - \dfrac{y}{2}$

$V = \pi\int_0^6 \left(3 - \dfrac{y}{2}\right)^2 dy = 18\pi$

f) the line $x = 4$

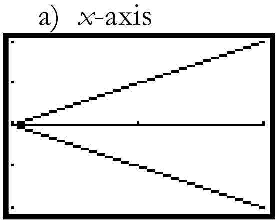


$R = 4 - \dfrac{y}{2}, r = 1$

$V = \pi\int_0^6 \left[\left(4 - \dfrac{y}{2}\right)^2 - 1\right] dy = 36\pi$

Example 2) Find the volume if the region enclosing $y = x - 1, y = 0, x = 3$ is rotated about the

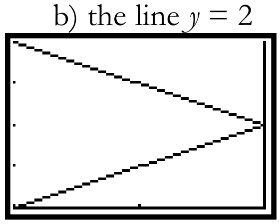
a) x-axis



$R = x - 1$

$V = \pi\int_1^3 (x - 1)^2 dx = 8\pi/3$

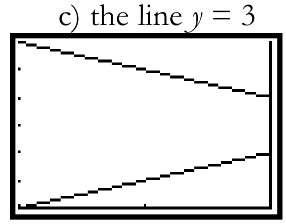
b) the line $y = 2$



$R = 2, r = 2 - (x - 1)$

$V = \pi\int_1^3 \left[4 - (3 - x)^2\right] dx = 16\pi/3$

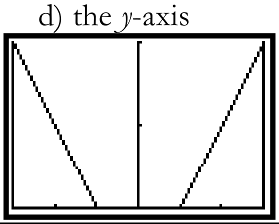
c) the line $y = 3$



$R = 3, r = 3 - (x - 1)$

$V = \pi\int_1^3 \left[9 - (4 - x)^2\right] dx = 28\pi/3$

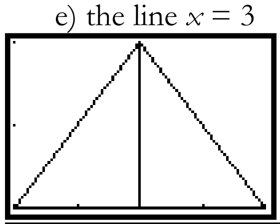
d) the y-axis



$R = 3, r = y + 1$

$V = \pi\int_0^2 \left[9 - (y + 1)^2\right] dy = 28\pi/3$

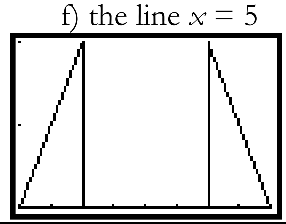
e) the line $x = 3$



$R = 2 - y$

$V = \pi\int_0^2 (2 - y)^2 dy = 8\pi/3$

f) the line $x = 5$

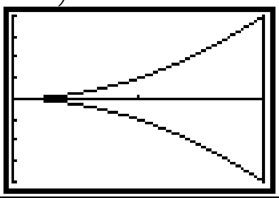


$R = 4 - y, r = 2$

$V = \pi\int_0^2 \left[(4 - y)^2 - 4\right] dy = 32\pi/3$

Example 3) Find the volume if the region enclosing $y = x^2, y = 0, x = 2$ is rotated about the

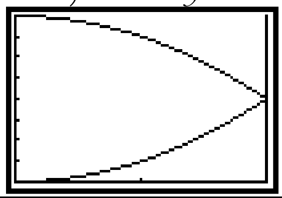
a) x-axis



$R = x^2$

$V = \pi \int_0^2 x^4 \, dx = 32\pi/5$

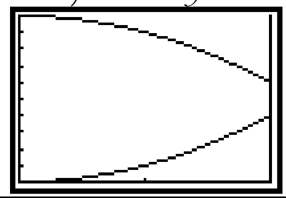
b) the line $y = 4$



$R = 4, r = 4 - x^2$

$V = \pi \int_0^2 \left[16 - \left(4 - x^2\right)^2\right] dx = 224\pi/15$

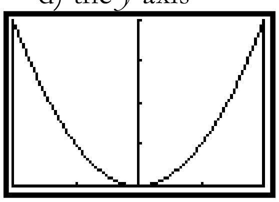
c) the line $y = 5$



$R = 5, r = 5 - x^2$

$V = \pi \int_0^2 \left[25 - \left(5 - x^2\right)^2\right] dx = 304\pi/15$

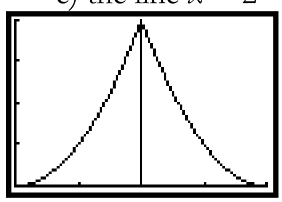
d) the y-axis



$R = 2, r = \sqrt{y}$

$V = \pi \int_0^4 (4 - y) \, dy = 8\pi$

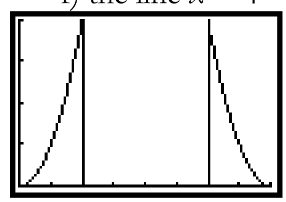
e) the line $x = 2$



$R = 2 - \sqrt{y}$

$V = \pi \int_0^4 \left(2 - \sqrt{y}\right)^2 dy = 8\pi/3$

f) the line $x = 4$

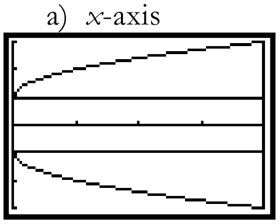


$R = 4 - \sqrt{y}, r = 2$

$V = \pi \int_0^4 \left[\left(4 - \sqrt{y}\right)^2 - 4\right] dy = 40\pi/3$

Example 4) Find the volume if the region enclosing $y = 1 + \sqrt{x}, y = 1, x = 4$ is rotated about the

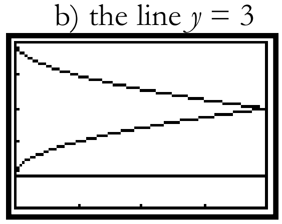
a) x-axis



$R = 1 + \sqrt{x}, r = 1$

$V = \pi \int_0^4 \left[\left(1 + \sqrt{x}\right)^2 - 1\right] dx = 56\pi/3$

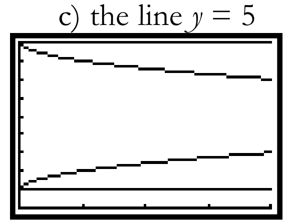
b) the line $y = 3$



$R = 2, r = 2 - \sqrt{x}$

$V = \pi \int_0^4 \left[4 - \left(2 - \sqrt{x}\right)^2\right] dx = 40\pi/3$

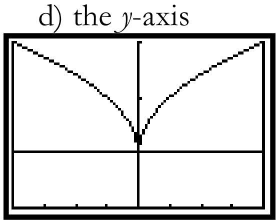
c) the line $y = 5$



$R = 4, r = 4 - \sqrt{x}$

$V = \pi \int_0^4 \left[16 - \left(4 - \sqrt{x}\right)^2\right] dx = 104\pi/3$

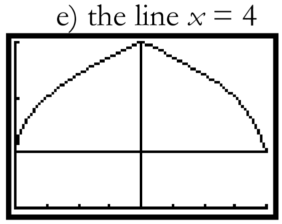
d) the y-axis



$R = 4, r = (y - 1)^2$

$V = \pi \int_1^3 \left[16 - (y - 1)^4\right] dy = 25.6\pi$

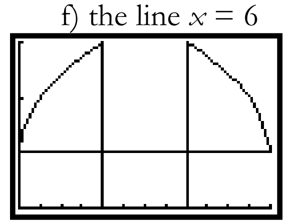
e) the line $x = 4$



$R = 4 - (y - 1)^2$

$V = \pi \int_1^3 \left[4 - (y - 1)^2\right]^2 dy = 17.067\pi$

f) the line $x = 6$

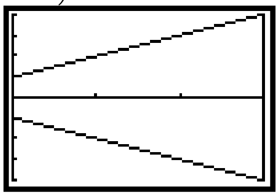


$R = 6 - (y - 1)^2, r = 2$

$V = \pi \int_1^3 \left[\left[6 - (y - 1)^2\right]^2 - 4\right] dy = 38.4\pi$

Example 5) Find the volume if the region enclosing $y = x + 1, x = 0, y = 0, x = 3$ is rotated about the

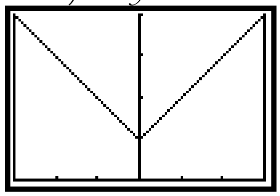
a) *x*-axis



$R = x + 1$

$V = \pi \int_0^3 (x+1)^2\,dx = 21\pi$

b) the *y*-axis

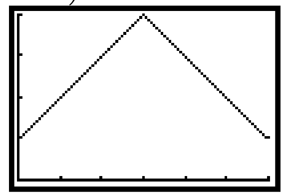


$R = 3, r = x = y - 1$

$V = \pi \int_0^1 9\,dy + \pi \int_1^4 \left[9 - (y-1)^2\right] dy$

$V = 27\pi$

c) the line *x* = 3



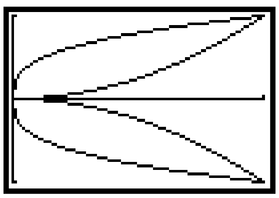
$R = 3, R = 3 - x = 4 - y$

$V = \pi \int_0^1 9\,dy + \pi \int_1^4 \left[(4-y)^2\right] dy$

$V = 18\pi$

Example 6) Find the volume if the region enclosing $y = x^2, y = \sqrt[3]{x}$ is rotated about the

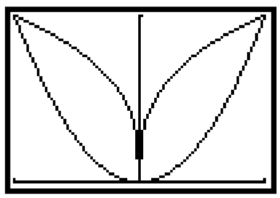
a) *x*-axis



$R = \sqrt[3]{x}, r = x^2$

$V = \pi \int_0^1 \left[x^{2/3} - x^4\right] dx = 0.4\pi$

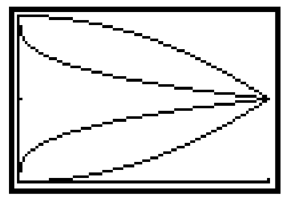
b) the *y*-axis



$R = \sqrt{y}, r = y^3$

$V = \pi \int_0^1 \left[y - y^6\right] dy = 0.357\pi$

c) the line *y* = 1

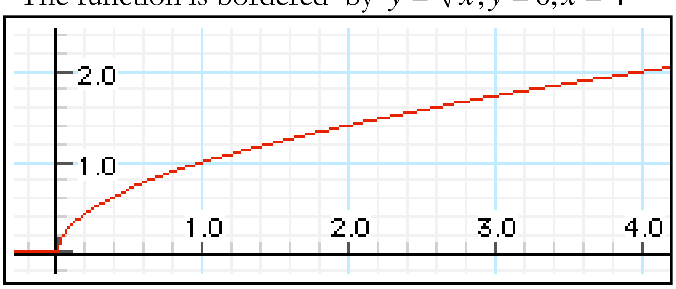


$R = 1 - x^2, r = 1 - \sqrt[3]{x},$

$V = \pi \int_0^1 \left[\left(1 - x^2\right)^2 - \left(1 - \sqrt[3]{x}\right)^2\right] dx = 0.433\pi$

Example 7) The region in the figure below, is revolved about the indicated lines. Order the volumes of the resulting solids from smallest to largest. Justify your answers. a) *x*-axis b) *y*-axis c) *x* = 4, d) *y* = 2

The function is bordered by $y = \sqrt{x}, y = 0, x = 4$



a: $V = \pi \int_0^4 x\,dx = 8\pi$

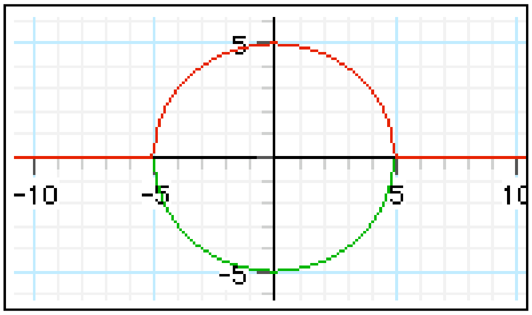
b: $V = \pi \int_0^2 \left(16 - y^4\right) dy = 25.6\pi$

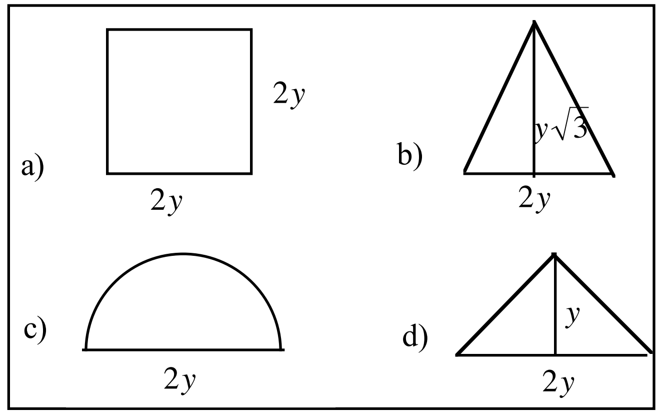
c: $V = \pi \int_0^2 \left(4 - y^2\right)^2 dy = 17.067\pi$

d: $V = \pi \int_0^4 \left[4 - \left(2 - \sqrt{x}\right)^2\right] dx = 13.333\pi$

a - d - c - b

Example 8) Find the volume of the solid whose base is bounded by the circle $x^2 + y^2 = 25$ with the indicated cross sections taken perpendicular to the x-axis.





a) squares

$$A = (2y)^2 \quad V = 8\int_0^5 \left(25 - x^2\right) dx = \frac{2000}{3}$$

b) equilateral triangles

$$A = \frac{1}{2}(2y)\left(y\sqrt{3}\right) \quad V = 2\sqrt{3}\int_0^5 \left(25 - x^2\right) dx = 288.675$$

c) semi-circles

$$A = \frac{1}{2}\pi y^2 \quad V = \pi\int_0^5 \left(25 - x^2\right) dx = 261.8$$

d) isosceles right triangles

$$A = \frac{1}{2}(2y)(y) \quad V = 2\int_0^5 \left(25 - x^2\right) dx = \frac{500}{3}$$

Example 9) Find the volume of the solid whose base is bounded by the lines $y = x - 4, y = 4 - x$ and $x = 0$ with the indicated cross sections taken perpendicular to the x-axis

a) squares

$$A = (2y)^2 \quad V = 4\int_0^4 (x-4)^2\, dx = \frac{256}{3}$$

b) equilateral triangles

$$A = \frac{1}{2}(2y)\left(y\sqrt{3}\right) \quad V = \sqrt{3}\int_0^4 (x-4)^2\, dx = 36.950$$

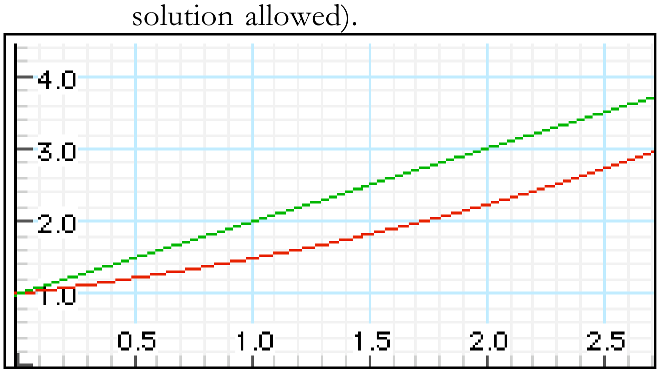
c) semi circles

$$A = \frac{1}{2}\pi y^2 \quad V = \frac{1}{2}\pi\int_0^4 (x-4)^2\, dx = 33.510$$

d) isosceles right triangles

$$A = \frac{1}{2}(2y)(y) \quad V = \int_0^4 (x-4)^2\, dx = \frac{64}{3}$$

Example 10) A horn for a public address system is to be made with the inside cross sections increasing exponentially with distance from the speaker. The horn will have the shape of the solid formed when the region bounded by $y = e^{0.4x}$ and $y = x + 1$ from $x = 0$ to $x = 3$ is rotated about the x-axis. If x and y are in feet, find the volume of the material used to make this speaker. (Calculator solution allowed).



$$V = \pi\int_0^3 \left[e^{.8x} - (x+1)^2\right] dx = 8.471\pi = 26.613 \text{ ft}^3$$

Volume by Disks/Washers - Homework

1. Find the volume if the region enclosing $y = 4 - x, x = 0, y = 0$ is rotated about the

a) x-axis

$R = y = 4 - x$

$V = \pi \int_0^4 (4 - x)^2 dx$

$V = \frac{64\pi}{3}$

b) the line $y = 4$

$R = 4 \quad r = 4 - y = 4 - (4 - x)$

$V = \pi \int_0^4 (16 - x^2)\, dx$

$V = \frac{128\pi}{3}$

c) the line $y = 5$

$R = 5 \quad r = 5 - y = 5 - (4 - x)$

$V = \pi \int_0^4 \left[25 - (x + 1)^2\right] dx$

$V = \frac{176\pi}{3}$

d) the y-axis

$R = x = 4 - y$

$V = \pi \int_0^4 (4 - y)^2 dy$

$V = \frac{64\pi}{3}$

e) the line $x = 4$

$R = 4 \quad r = 4 - x = 4 - (4 - y)$

$V = \pi \int_0^4 (16 - y^2)\, dy$

$V = \frac{128\pi}{3}$

f) the line $x = 6$

$R = 6 \quad r = 6 - x = 6 - (4 - y)$

$V = \pi \int_0^4 \left[36 - (y + 2)^2\right] dy$

$V = \frac{224\pi}{3}$

2. Find the volume if the region enclosing $y = x^2 + x, y = 0, x = 2$ is rotated about the

a) x-axis

$R = y = x^2 + x$

$V = \pi \int_0^2 (x^2 + x)^2 dx$

$V = \frac{256\pi}{15}$

b) the line $y = 6$

$R = 6 \quad r = 6 - y = 6 - (x^2 + x)$

$V = \pi \int_0^2 \left[36 - (6 - x^2 - x)^2\right] dx$

$V = \frac{584\pi}{15}$

c) the line $y = 9$

$R = 9 \quad r = 9 - y = 9 - (x^2 + x)$

$V = \pi \int_0^2 \left[81 - (9 - x^2 - x)^2\right] dx$

$V = \frac{1004\pi}{15}$

3. Find the volume if the region enclosing $y = x^3, x = 0, y = 8$ is rotated about the

a) x-axis

$R = 8 \quad r = y = x^3$

$V = \pi \int_0^2 \left(64 - x^6\right) dx$

$V = \dfrac{768\pi}{7}$

b) the line $y = 8$

$R = 8 - y = 8 - x^3$

$V = \pi \int_0^2 \left(8 - x^3\right)^2 dx$

$V = \dfrac{576\pi}{7}$

c) the line $y = 9$

$R = 9 - y = 9 - x^3 \quad r = 1$

$V = \pi \int_0^2 \left[\left(9 - x^3\right)^2 - 1\right] dx$

$V = \dfrac{744\pi}{7}$

d) the y-axis

$R = x = \sqrt[3]{y}$

$V = \pi \int_0^8 y^{2/3}\, dy$

$V = 19.2\pi$

e) the line $x = 2$

$R = 2 \quad r = 2 - x = 2 - \sqrt[3]{y}$

$V = \pi \int_0^8 \left[4 - \left(2 - y^{1/3}\right)^2\right] dy$

$V = 28.8\pi$

f) the line $x = 3$

$R = 3 \quad r = 3 - x = 3 - \sqrt[3]{y}$

$V = \pi \int_0^8 \left[9 - \left(3 - y^{1/3}\right)^2\right] dy$

$V = 52.8\pi$

4. Find the volume if the region enclosing $y = x^2, y = 2x, x \geq 0$ is rotated about the

a) x-axis

$R = y = 2x \quad r = y = x^2$

$V = \pi \int_0^2 \left(4x^2 - x^4\right) dx$

$V = \dfrac{64\pi}{15}$

b) the line $y = 4$

$R = 4 - y = 4 - x^2 \quad r = 4 - y = 4 - 2x$

$V = \pi \int_0^2 \left[\left(4 - x^2\right)^2 - \left(4 - 2x\right)^2\right] dx$

$V = \dfrac{32\pi}{5}$

c) the line $y = 7$

$R = 7 - y = 7 - x^2 \quad r = 7 - y = 7 - 2x$

$V = \pi \int_0^2 \left[\left(7 - x^2\right)^2 - \left(7 - 2x\right)^2\right] dx$

$V = \dfrac{72\pi}{5}$

d) the y-axis

$R = x = \sqrt{y} \quad r = x = \dfrac{y}{2}$

$V = \pi \int_0^4 \left[y - \dfrac{y^2}{4}\right] dy$

$V = \dfrac{8\pi}{3}$

e) the line $x = 2$

$R = 2 - x = 2 - \dfrac{y}{2} \quad r = 2 - x = 2 - \sqrt{y}$

$V = \pi \int_0^4 \left[\left(2 - \dfrac{y}{2}\right)^2 - \left(2 - \sqrt{y}\right)^2\right] dy$

$V = 2.667\pi$

f) the line $x = 3$

$R = 3 - x = 3 - \dfrac{y}{2} \quad r = 3 - x = 3 - \sqrt{y}$

$V = \pi \int_0^4 \left[\left(3 - \dfrac{y}{2}\right)^2 - \left(3 - \sqrt{y}\right)^2\right] dy$

$V = 5.333\pi$

5. Find the volume if the region enclosing $y = 1 + \sqrt{x}, x = 0, y = 0, x = 9$ is rotated about the

a) x-axis

$R = y = 1 + \sqrt{x}$

$V = \pi \int_0^9 \left(1 + \sqrt{x}\right)^2 dx$

$V = 85.5\pi$

b) the line $y = 4$

$R = 4 \quad r = 4 - y = 4 - \left(1 + \sqrt{x}\right)$

$V = \pi \int_0^9 \left[16 - \left(3 - \sqrt{x}\right)^2\right] dx$

$V = 130.5\pi$

c) the line $y = 5$

$R = 5 \quad r = 5 - y = 5 - \left(1 + \sqrt{x}\right)$

$V = \pi \int_0^9 \left[25 - \left(4 - \sqrt{x}\right)^2\right] dx$

$V = 184.5\pi$

d) the y-axis

Disks	Washers
$R = 9$	$R = 9 \quad r = x = (y-1)^2$
$V = \pi \int_0^1 81\, dy$ +	$\pi \int_1^4 \left(81 - (y-1)^4\right) dy$

$V = 81\pi + 194.4\pi = 275.4\pi$

e) the line $x = 9$

Disks	Disks
$R = 9$	$R = 9 - x = 9 - (y-1)^2$
$V = \pi \int_0^1 81\, dy$ +	$\pi \int_1^4 \left(9 - (y-1)^2\right)^2 dy$

$V = 81\pi + 129.6\pi = 210.6\pi$

6. Find the volume if the first quadrant region $y = \sin x$ and $y = \cos x$ on $[0, \pi/4]$ is rotated about

a) x-axis

$R = y = \cos x \quad r = y = \sin x$

$V = \pi \int_0^{\pi/4} \left(\cos^2 x - \sin^2 x\right) dx$

$V = \frac{\pi}{2}$

b) the y-axis

$R = x = \sin^{-1} y \quad R = x = \cos^{-1} y$

$V = \pi \int_0^{\sqrt{2}/2} \left(\sin^{-1} y\right)^2 dy + \pi \int_{\sqrt{2}/2}^1 \left(\cos^{-1} y\right)^2 dy$

$V = .133\pi + .089\pi = .222\pi$

7. A tank on the wing of a jet plane is formed by revolving the region bounded by the graph of $y = \frac{1}{10} x^2 \sqrt{3 - x}$ and the x-axis about the x-axis where x and y are measured in meters. Find the volume of the tank.

$R = \frac{x^2\sqrt{3-x}}{10} \qquad V = \pi \int_0^3 \left(\frac{x^2\sqrt{3-x}}{10}\right)^2 = \frac{\pi}{100} \int_0^3 \left[x^4(3-x)\right] dx = .243\pi = 0.763\text{m}^3$

8. The region bounded by the curve $y=\sqrt{x}, x=0, y=0, x=9$ is rotated about the *x*-axis.

a) Find the value of *a* in the interval [0, 9] that divides the region into 2 parts of equal area.

$$2\int_0^a x^{1/2}dx = \int_0^9 x^{1/2}dx$$
$$\frac{4}{3}a^{3/2} = 18$$
$$a^{3/2} = \frac{27}{2}$$
$$a = 5.700$$

b) Find the value of *a* in the interval [0, 9] that divides the solid into 2 parts of equal volume.

$$2\pi\int_0^a x\,dx = \pi\int_0^9 x\,dx$$
$$a^2 = \frac{81}{2}$$
$$a = \frac{9}{\sqrt{2}}$$
$$a = 6.364$$

9. Find the volume of the solid whose base is bounded by the circle whose center is the origin and whose radius is 10 with the indicated cross sections perpendicular to the *x*-axis

a) squares

$$A=(2y)^2 \quad V = 8\int_0^{10}\left(100-x^2\right)dx = \frac{16000}{3}$$

b) equilateral triangles

$$A=\frac{1}{2}(2y)\left(y\sqrt{3}\right) \quad V = 2\sqrt{3}\int_0^{10}\left(100-x^2\right)dx = 2309.401$$

c) semi-circles

$$A=\frac{1}{2}\pi y^2 \quad V = \pi\int_0^{10}\left(100-x^2\right)dx = 2094.395$$

d) isosceles right triangles

$$A=\frac{1}{2}(2y)(y) \quad V = 2\int_0^{10}\left(100-x^2\right)dx = \frac{4000}{3}$$

10. Find the volume of the solid whose base is bounded by the curves $y=x^2-x-3$ and $y=x$ with the indicated cross sections taken perpendicular to the *x*-axis

$$d = x^2-x-3-x = x^2-2x-3$$

a) squares

$$A=d^2$$
$$\int_{-1}^{3}\left(x^2-2x-3\right)^2dx = \frac{512}{15}$$

b) equilateral triangles

$$A=\frac{1}{2}d\left(\frac{d\sqrt{3}}{2}\right) = \frac{d^2\sqrt{3}}{4}$$
$$\frac{\sqrt{3}}{4}\int_{-1}^{3}\left(x^2-2x-3\right)^2dx = 14.780$$

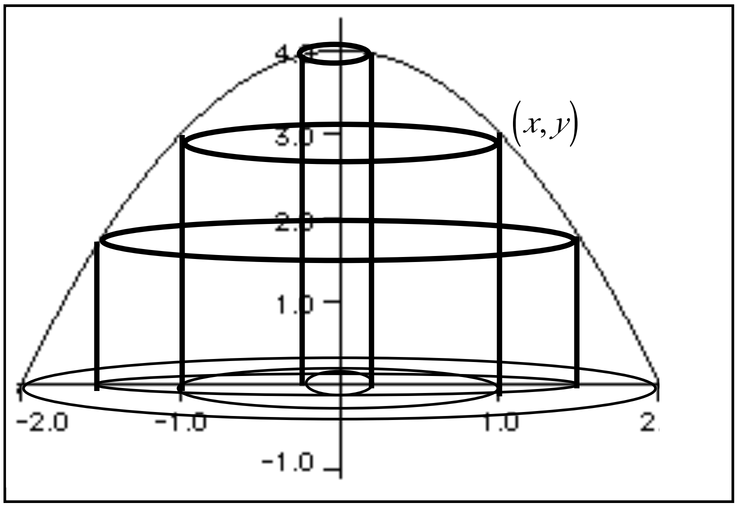
c) semi-circles

$$A=\frac{1}{2}\pi\left(\frac{d}{2}\right)^2 = \frac{\pi d^2}{8}$$
$$\frac{\pi}{8}\int_{-1}^{3}\left(x^2-2x-3\right)^2dx = 13.404$$

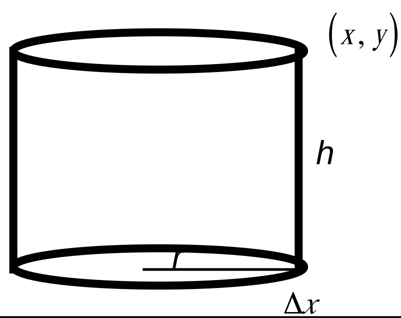
d) isosceles right triangles

$$A=\frac{1}{2}d\left(\frac{d}{2}\right) = \frac{d^2}{4}$$
$$\frac{1}{4}\int_{-1}^{3}\left(x^2-2x-3\right)^2dx = \frac{128}{15}$$

Volume by Cylindrical Shells - Classwork



There is another way to determine the volume of a curve that is rotated about an axis - the method of cylindrical shells. Let us take some function $y = f(x)$ and rotate it about the *y*-axis as shown above. Instead of creating a disk, we will create a cylinder. This cylinder is very thin - its sides are like paper. We call it a cylindrical shell. We can draw many such cylinders (above three are drawn). Note that they all have different heights and different radii.



The surface area of this shell is
$2\pi(\text{radius})(\text{height})$
Since this shell has thickness Δx
The volume of the shell can be expressed
$V = 2\pi(\text{radius})(\text{height})\Delta x$

Since we are using infinitely thin shells $\left(\lim_{\Delta x \to 0}\right)$, we can say that the volume of our rotated region is

$$V = \lim_{n\to\infty}\left[\sum_{i=1}^{n} 2\pi \cdot (\text{radius}_i)(\text{height}_i)\Delta x\right]$$

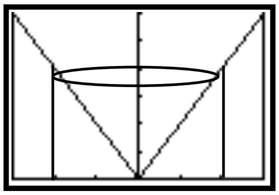
$\int_{x=a}^{x=b} 2\pi r \cdot h\, dx$ if rotated about *y*-axis or vertical line

$\int_{y=a}^{y=b} 2\pi r \cdot h\, dy$ if rotated about *x*-axis or horizontal line

So when you attempt a problem by cylindrical shells, your job is to determine both the radius and height. Note that in using the disk/washer method, it is usually better to rotate about the *x*-axis or lines parallel to the *x*-axis (*y* = constant) because the outside and inside radii are expressed as a vertical distance which will be a function of *x*. The shell method's advantage is usually in rotating about the *y*-axis or lines parallel to the *y*-axis (*x* = constant) because the radius is a horizontal distance given some function of *x* and the height is the vertical distance which is a function of *x*. So from now on, it is recommended that you use shells when rotating around the *y*-axis or lines parallel to the *y*-axis.

1. Find the volume if the region enclosing $y = 2x, y = 0,\ x = 3$ is rotated about the

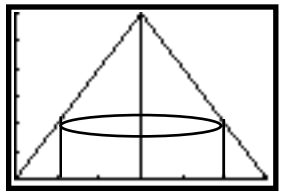
a) the *y*-axis



$r = x, h = 2x$

$V = 2\pi\int_0^3 2x^2 dx = 36\pi$

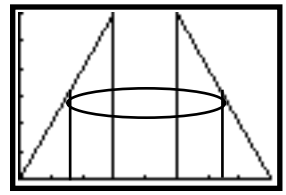
b) the line *x* = 3



$r = 3 - x, h = 2x$

$V = 2\pi\int_0^3 2x(3 - x)\,dx = 18\pi$

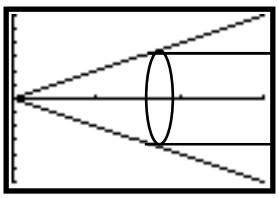
c) the line *x* = 4



$r = 4 - x, h = 2x$

$V = 2\pi\int_0^3 2x(4 - x)\,dx = 36\pi$

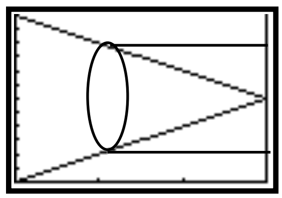
d) *x*-axis



$r = y, h = 3 - \frac{y}{2}$

$V = 2\pi\int_0^6 y\left(3 - \frac{y}{2}\right) dy = 36\pi$

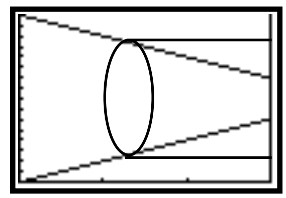
e) the line *y* = 6



$r = 6 - y, h = 3 - \frac{y}{2}$

$V = 2\pi\int_0^6 (6 - y)\left(3 - \frac{y}{2}\right) dy = 72\pi$

f) the line *y* = 8



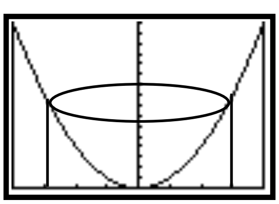
$r = 8 - y, h = 3 - \frac{y}{2}$

$V = 2\pi\int_0^6 (8 - y)\left(3 - \frac{y}{2}\right) dy = 108\pi$

Again, the shell method's greatest advantage is rotating around the *y*-axis when the problem can be in terms of *x*. A downside of the shell method is that you may not be able to integrate using the Fundamental Theorem. Calculators are of a real advantage here.

2. Find the volume if the region enclosing $y = x^2, y = 0, x = 4$ is rotated about the

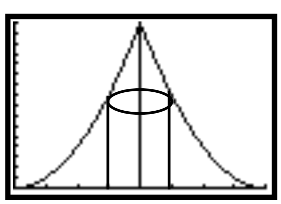
a) *y*-axis



$r = x, h = x^2$

$V = 2\pi\int_0^4 x^3 dx = 128\pi$

b) the line *x* = 4

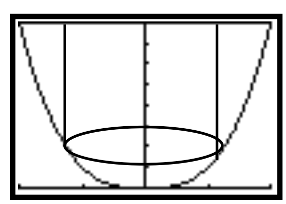


$r = 4 - x, h = x^2$

$V = 2\pi\int_0^4 (4 - x)x^2 dx = \frac{128}{3}\pi$

3. Find the volume if the region enclosing $y = x^3, y = 8, x = 0$ is rotated about the

a) *y*-axis



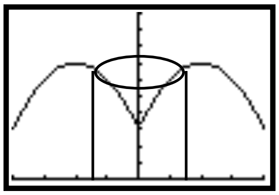
$r = x, h = 8 - x^3$

$V = 2\pi\int_0^2 x\left(8 - x^3\right)dx = \frac{96}{5}\pi$

Volume by Cylindrical Shells - Homework

1. Find the volume if the region enclosing $y = -x^2 + 4x + 3, x = 0, x = 4, y = 0$ is rotated about

a) y-axis

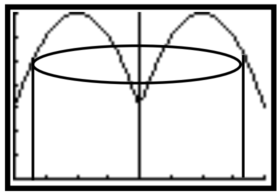


$r = x, h = -x^2 + 4x + 3$

$V = 2\pi\int_0^4 x\left(-x^2 + 4x + 3\right)dx$

$V = \frac{272}{3}\pi$

b) the line $x = 4$

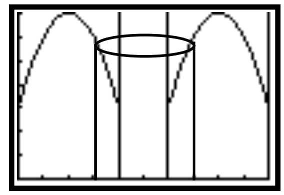


$r = 4 - x, h = -x^2 + 4x + 3$

$V = 2\pi\int_0^4 (4 - x)\left(-x^2 + 4x + 3\right)dx$

$V = \frac{272}{3}\pi$

c) the line $x = 5$



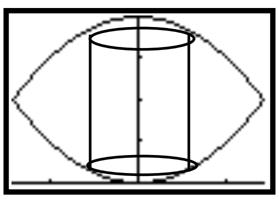
$r = 4 - x, h = -x^2 + 4x + 3$

$V = 2\pi\int_0^4 (5 - x)\left(-x^2 + 4x + 3\right)dx$

$V = 136\pi$

2. Find the volume if the first quadrant region enclosing $y = x^2, y = 8 - x^2$ is rotated about

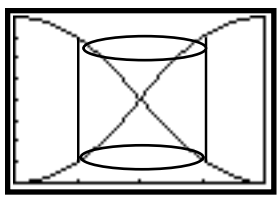
a) y-axis



$r = x, h = 8 - 2x^2$

$V = 2\pi\int_0^2 x\left(8 - 2x^2\right)\,dx = 16\pi$

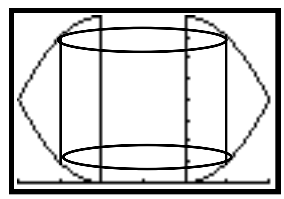
b) the line $x = 2$



$r = 2 - x, h = 8 - 2x^2$

$V = 2\pi\int_0^2 (2 - x)\left(8 - 2x^2\right)\,dx = \frac{80}{3}\pi$

c) the line $x = -1$

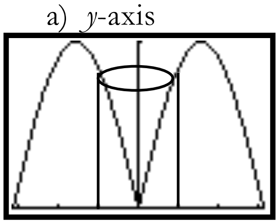


$r = x + 1, h = 8 - 2x^2$

$V = 2\pi\int_0^2 (x + 1)\left(8 - 2x^2\right)\,dx = \frac{112}{3}\pi$

3. Find the volume if the region enclosed by $y = \sin x, y = 0$ on $[0, \pi]$ is rotated about

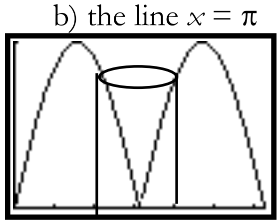
a) y-axis



$r = x, h = \sin x$

$V = 2\pi\int_0^\pi x\sin x\,dx = 2\pi^2$

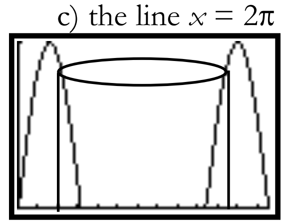
b) the line $x = \pi$



$r = \pi - x, h = \sin x$

$V = 2\pi\int_0^\pi (\pi - x)\sin x\,dx = 2\pi^2$

c) the line $x = 2\pi$



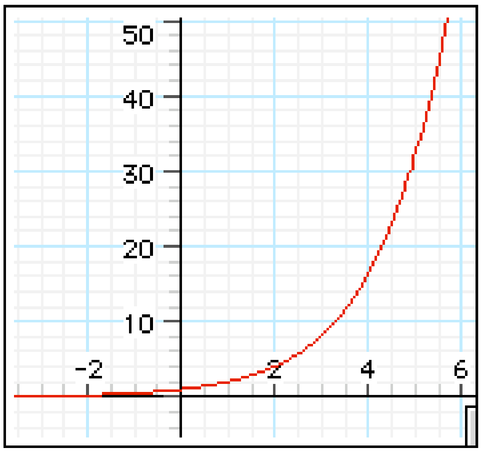
$r = 2\pi - x, h = \sin x$

$V = 2\pi\int_0^\pi (2\pi - x)\sin x\,dx = 6\pi^2$

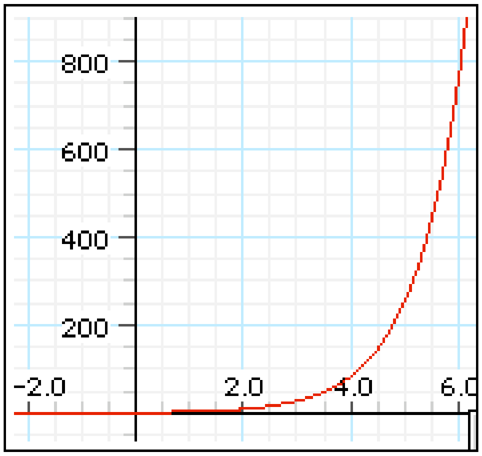
Review of Exponentials and Logarithms - Classwork

In our study of calculus, we have examined derivatives and integrals of polynomial expressions, rational expressions, and trignometric expressions. What we have not examined are exponential expressions, expressions of the form $y = a^x$. While these are covered extensively in precalculus, a little review is in order and these types of expressions are very prevalent in the calculus theatre.

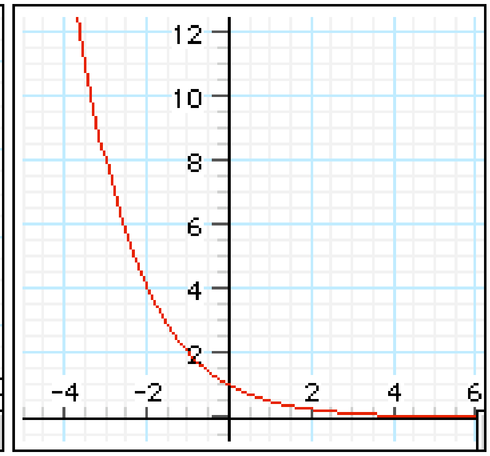
An expressions in the form of $y = a^x$ will graph an exponential. An exponential graph tends to "explode" based on the value of *x*, since the *x* is in the exponent.



$y = 2^x$



$y = 3^x$



$y = .5^x$

The graph of various exponential curves are shown above. We know that the graph of $y = 1^x$ graphs **horiz. line**
If $a < 0$, the graph of $y = a^x$ will not exist at certain points for instance, what is $(-2)^{1/2}$? **DNE**
So it only makes sense to examine functions in form of $y = a^x$, if $a > 0$, $a \neq 1$. When $a > 1$, we get what is called a growth curve and the larger *a* is, the steeper the growth curve is. If $0 < a < 1$, the we get a decay cuvre as shown in the 3rd graph above. No matter what, exponential curves in the form of $y = a^x$ have certain features.

What point do they have in common? $(0,1)$ What is the domain? $(-\infty,\infty)$ What is the range? $(0,\infty)$

Solving basic exponential equations can be accomplished by using the fact that if $a^x = a^y$, then $x = y$.

Examples) Solve for *x*.

1) $2^{x+1} = 8$
$2^{x+1} = 2^3$
$x = 2$

2) $3^{2x-3} = \frac{1}{3}$
$3^{2x-3} = 3^{-1}$
$x = 1$

3) $4^{5x-1} = \sqrt[3]{32}$
$(2^2)^{5x-1} = 2^{5/3}$
$2^{10x-2} = 2^{5/3}$
$x = \frac{11}{30}$

4) $7^{2x+3} = \left(\frac{1}{49}\right)^{3-x}$
$7^{2x+3} = 7^{2x-6}$
No Solution

5) $8^{5-2x} = 1$
$8^{5-2x} = 8^0$
$x = \frac{5}{2}$

Solving exponential equations like the ones above are easy when each side of the equation have common bases. But problems like $3^{x-1} = 4$ cause problems. With that problem created, we introduced the concept of logarithms. A logarithm is simply an inverse of an exponential. Students typically hear the word logarithm and go into a cold sweat because they do not understand them. So lets get it straight once and for all.

The statement $y = b^x$ can be written in an alternate way: $x = \log_b y$. They mean the same thing.

Whenever you are given a logarithmic statement, write it exponentially. You will know the answer!

Examples: find the value of the following:

1) $\boxed{\log_2 8 = 3}$ 2) $\boxed{\log_5 \frac{1}{25} = -2}$ 3) $\boxed{\log_8 \sqrt{2} = \frac{1}{6}}$ 4) $\boxed{5\log_6 1 = 0}$ 5) $\boxed{\log_7 0 \text{ DNE}}$

If the base is not specified, it is assumed to be 10. $\log_{10} x$ and $\log x$ are the same things.

Examples) Find the value of the following:

6) $\boxed{\log 100 = 2}$ 7) $\boxed{\log 1 = 0}$ 8) $\boxed{5\log\sqrt{10} = \frac{5}{2}}$ 9) $\boxed{\log\frac{1}{1000} = -3}$ 10) $\boxed{\log 10^{\sqrt{5}} = \sqrt{5}}$

On what seems to be a side note, let's examine the expression $y = \left(1 + \frac{1}{x}\right)^x$ for various values of *x*. What we are doing is looking at $\left(1+\frac{1}{1}\right)^1, \left(1+\frac{1}{2}\right)^2, \left(1+\frac{1}{3}\right)^3, \left(1+\frac{1}{4}\right)^4 \ldots \left(1+\frac{1}{50}\right)^{50} \ldots$ Logic would tell us that as *x* gets larger, $1 + \frac{1}{x}$ gets $\boxed{\text{closer to } 0}$ and this the limit as *x* approaches infinity of $1 + \frac{1}{x}$ is $\boxed{1}$. Thus, logic also dictates that $\lim_{x \to \infty}\left(1 + \frac{1}{x}\right)^x = \boxed{1}$. But when you play with infinity, logic doesn't always work. You can see that if you set up your calculator with the expression Y1 = $\left(1 + \frac{1}{x}\right)^x$ and look at a table of values.

x	$\left(1+\frac{1}{x}\right)^x$		x	$\left(1+\frac{1}{x}\right)^x$		x	$\left(1+\frac{1}{x}\right)^x$		x	$\left(1+\frac{1}{x}\right)^x$
1	2		100	2.70481383		1000	2.71692393		100000	2.71826824
2	2.25		110	2.70602808		2000	2.71760257		200000	2.71827503
3	2.37037037		120	2.70704149		3000	2.71782892		300000	2.7182773
4	2.44140625		130	2.70790008		4000	2.71794212		400000	2.71827843
5	2.48832		140	2.70863681		5000	2.71801005		500000	2.71827911
6	2.52162637		150	2.70927591		6000	2.71805534		600000	2.71827956
7	2.5464997		160	2.70983558		7000	2.71808769		700000	2.71827989
8	2.56578451		170	2.71032975		8000	2.71811196		800000	2.71828013
9	2.58117479		180	2.7107693		9000	2.71813083		900000	2.71828032
10	2.59374246		190	2.71116279		10000	2.71814593		1000000	2.71828047

If should be obvious that as *x* gets larger and larger, the expression $\left(1+\frac{1}{x}\right)^x$ is not approaching one but the number **2.718**. This number is a very special number in mathematics and is called Euler's number. Leonhard Euler (1707 - 1783) discovered this number and it is known as *e*. The value of *e* is 2.718281828…. *e* is a number which, like π and $\sqrt{2}$, continues on forever without any pattern. (Note: the 1828 in *e*, although appearing twice consecutively near the start does not appear again for a very long while. It is completely coincidental that it appears twice)

The number *e* is such an important number (if you would have to decide what the 5 most important numbers are, what would they be? $\boxed{(0,1,\pi,i,e)}$), that it forms the basic of what are called natural logarithms of Napierian logs (after John Napier, 1550-1617, who first used them). Just as logarithms (log) use base 10, natural logs (ln) use base *e*. When you wish to find the value of a log, you write the expression exponentially. You do the same thing with a natural log except that your base is now *e*.

For instance, to find ln 10, you call it *x*, and are now solving the equation $e^x = 10$. Since *e* is slightly below 3, we expect ln 10 to be between the values of $\boxed{\text{2 and 3}}$.

So, given the function $y = e^x$, the domain is $\boxed{(-\infty.\infty)}$ and the range is $\boxed{(0,\infty)}$.

Examples) Find the value of the following:

11) $\boxed{\ln e^4 = 4}$ 12) $\boxed{9\ln 1 = 0}$ 13) $\boxed{8\ln\sqrt{e} = 4}$ 14) $\boxed{\ln\frac{1}{e^3} = -3}$ 15) $\boxed{e^{\ln 5} = 5}$

There are three basic rules for operation with logarithms that you must know. They are as follows:

$$\boxed{1.\ \log(a\cdot b) = \log a + \log b \qquad 2.\ \log\left(\frac{a}{b}\right) = \log a - \log b \qquad 3.\ \log a^b = b\log a}$$

These rules work with logs to any base of the ln function.

Examples) Find the value of the following expressions:

16) $\log 25 + \log 4$
$\log 100$
2

17) $\log_2 40 - \log_2 5$
$\log_2 8$
3

18) $\log 10^{35}$
35

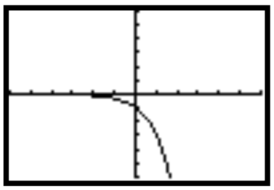
19) $\log_4 x + \log_4(x+12) = 3$
$\log_4 x(x+12) = 3$
$64 = x(x+12)$
$(x-4)(x+16) = 0$
$x = 4$

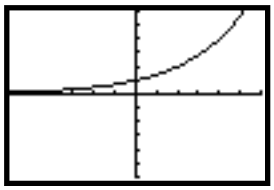
\- solve for *x*

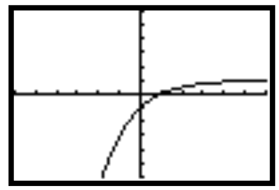
Review of Exponentials and Logarithms - Homework

For each curve below, identify it by the proper equation letter. No calculators.

a. $y = 2^x$
b. $y = 2^{-x}$
c. $y = 2\left(2^x\right)$
d. $y = -3^x$
e. $y = -3\left(2^x\right)$
f. $y = .5\left(4^x\right)$
g. $y = 4^x - 2$
h. $y = .5^x - 3$
i. $y = -2\left(.5^x\right) + 1$
j. $y = 3^{x+1}$
k. $y = 2^{x-1} + 2$
l. $y = .5^{x-2}$
m. $y = 2^{x/2}$
n. $y = 1.1^x$
o. $y = 4^x + 4^{-x}$

1. 

2. 

3. 

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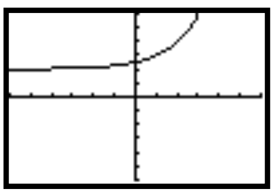
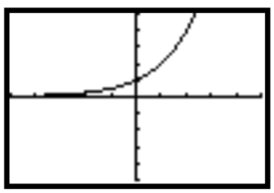
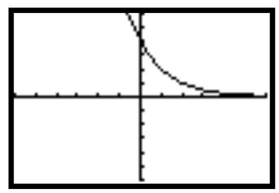
4. 

5. 

6. 

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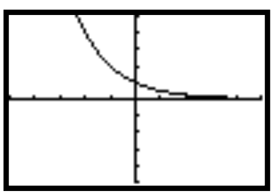
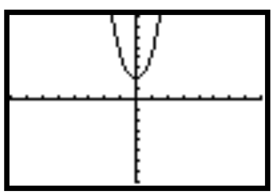
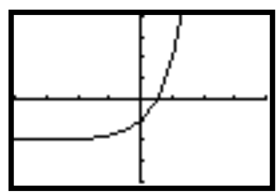
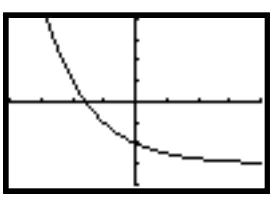
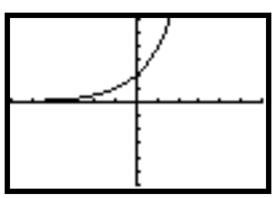
7. 

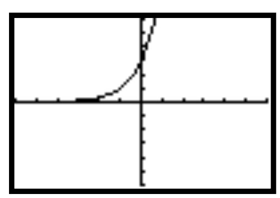
8. 

9. 

b	o	g

10. 

11. 

12. 

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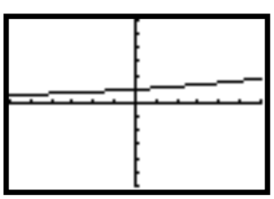
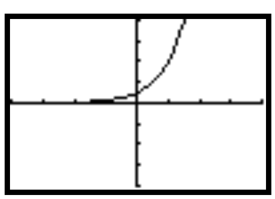
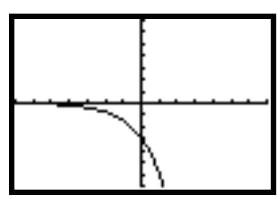
13. 

14. 

15. 

n	f	e

Solve for x

16. $2^{x-3}=16$
$x-3=4$
$x=7$

17. $3^{2x-3}=81$
$2x-3=4$
$x=\frac{7}{2}$

18. $5^{3x-3}=1$
$3x-3=0$
$x=1$

19. $2^{5-2x}=\frac{1}{2}$
$5-2x=-1$
$x=3$

20. $10^{5x+6}=\frac{1}{100}$
$5x+6=-2$
$x=\frac{-8}{5}$

21. $2^{4x+1}=\sqrt{2}$
$4x+1=\frac{1}{2}$
$x=\frac{-1}{8}$

22. $27^{3x+3}=9$
$3(3x+3)=2$
$x=\frac{-7}{9}$

23. $16^{x-3}=8^{x-3}$
$4(x-3)=3(x-3)$
$x=3$

24. $25^{6-2x}=\sqrt{5}$
$2(6-2x)=\frac{1}{2}$
$x=\frac{23}{8}$

25. $9^{2x-4}=\left(\frac{1}{27}\right)^{x-3}$
$2(2x-4)=-3(x-3)$
$x=\frac{17}{7}$

26. $\left(\frac{1}{32}\right)^{x+6}=\left(\frac{1}{8}\right)^{x-2}$
$-5(x+6)=-3(x-2)$
$x=-18$

27. $\left(\frac{1}{4}\right)^{2-2x}=\left(\sqrt[3]{2}\right)^{3x+6}$
$-2(2-2x)=\frac{1}{3}(3x+6)$
$x=2$

28. $\log_4 256=4$

29. $\log_2 8$

30. $\log_8 2=\frac{1}{3}$

31. $\log_5 125=3$

32. $\log_9 27=\frac{3}{2}$

33. $\log_7 1=0$

34. $\log_{25} 5=\frac{1}{2}$

35. $\log_8 16=\frac{4}{3}$

36. $\log_{\sqrt{3}} 27=6$

37. $\log_{\frac{1}{5}} \frac{1}{125} = 3$

38. $-5\log_{12} 12 = -5$

39. $10^{\log 29} = 29$

40. $\ln e^4 = 4$

41. $e^{\ln\sqrt{e}} = \sqrt{e}$

42. $\frac{3\log 10}{2\ln e} = \frac{3}{2}$

Solve each equation in terms of *x*.

43.
$\log_3(2x-2) = 2$
$9 = 2x - 2$
$x = \frac{11}{2}$

44.
$\log_7(7x-6) = 2$
$7x - 6 = 49$
$x = \frac{55}{7}$

45.
$\ln 3x + \ln 3 = 3$
$\ln 9x = 3$
$9x = e^3$
$x = \frac{e^3}{9}$

46.
$\log_5(x+3) - \log_5 x = 2$
$\log_5 \frac{(x+3)}{x} = 2$
$\frac{(x+3)}{x} = 25$
$x = \frac{1}{8}$

If $a = \log_2 6$ and $b = \log_2 10$, express the following in terms of *a* and *b*.

47.
$\log_2 24$
$\log_2(4 \cdot 6)$
$\log_2 4 + \log_2 6$
$2 + a$

48.
$\log_2 600$
$\log_2(100 \cdot 6)$
$\log_2 100 + \log_2 6$
$2b + a$

49.
$\log_2 \sqrt[4]{10}$
$\log_2(10)^{1/4}$
$\frac{1}{4}\log_2 10$
$\frac{b}{4}$

Differentiation of the Natural Log Function - Classwork

We have examined derivatives using the power rule, product rule, quotient rule, and trig. But what about the derivative of $y = \ln x$? We have no rule to cover such functions. Let's see how your calculator deals with it. In your TI-84 calculator, let Y1 = nDerviv(ln*x*,*x*,*x*) and set up your table with *x* starting at 1 and $\Delta x = 1$.

x	0	1	2	3	4	5	6	7	8	9	10
y'	ERROR	1.000	0.500	0.333	0.250	0.200	0.167	0.143	0.125	0.111	0.100

As you look at the relationship between the value of *x* and the value of the derivative of $\ln x$, it should be clear. What is it? The derivative is the reciprocal of x.

So we have a new differentiation rule - the ln rule:

$$\frac{d}{dx}[\ln x] = \frac{1}{x}, x > 0 \qquad \frac{d}{dx}[\ln u] = \frac{1}{u}\cdot\frac{du}{dx} = \frac{u'}{u}, u > 0 \qquad \frac{d}{dx}[\ln|u|] = \frac{1}{u}\cdot\frac{du}{dx} = \frac{u'}{u}, u \neq 0$$

What this says is to take the derivative of the ln of some expressions, you simply use the reciprocal of the expression multipled by the derivative of that expression. The expression must be a positive number.

Examples) Find the derivative of the following expressions:

1) $y = \ln(4x)$

$y' = \frac{1}{4x}\cdot 4 = \frac{1}{x}$

2) $y = \ln(x^2 - 3)$

$y' = \frac{2x}{x^2 - 3}$

3) $y = \ln(3x^2 - 5x + 8)$

$y' = \frac{6x - 5}{3x^2 - 5x + 8}$

4) $y = \ln\sqrt{x}$

$y' = \frac{\frac{1}{2}x^{-1/2}}{x^{1/2}} = \frac{1}{2x}$

Note: #4 - Better way: $y = \ln\sqrt{x} \Rightarrow y = \ln x^{1/2} \Rightarrow y = \frac{1}{2}\ln x \Rightarrow y' = \frac{1}{2x}$

You now have 5 sets of rules - power, product, quotient, trig (6 rules), and now "ln". Just because there is an "ln" in the problem does not mean it uses the ln rule above.

Examples) Find the derivative of the following expressions:

5) $y = x^2 \ln x$

$y' = x^2\left(\frac{1}{x}\right) + \ln x(2x)$

$y' = x + 2x\ln x$

6) $y = \frac{\ln x}{x}$

$y' = \frac{x(1/x) - \ln x}{x^2}$

$y' = \frac{1 - \ln x}{x^2}$

7) $y = \frac{x}{\ln x}$

$y' = \frac{\ln x - x}{(\ln x)^2}$

8) $y = (\ln x)^5$

$y' = \frac{5(\ln x)^4}{x}$

9) $y = \sqrt{\ln x} = (\ln x)^{1/2}$

$y' = \frac{1}{2}(\ln x)^{-1/2}\left(\frac{1}{x}\right)$

$y' = \frac{1}{2x(\ln x)^{1/2}}$

10) $y = \cos(\ln x)$

$y' = -\sin(\ln x)\left(\frac{1}{x}\right)$

$y' = \frac{-\sin(\ln x)}{x}$

11) $y = \ln(\cos x)$

$y' = \left(\frac{1}{\cos x}\right)(-\sin x)$

$y' = -\tan x$

12) $y = \ln(\ln x)$

$y' = \left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right)$

$y' = \frac{1}{x\ln x}$

Remember your log rules. They can help you to take derivative of harder expressions.

1. $\log(a\cdot b)=\log a+\log b$	2. $\log\left(\frac{a}{b}\right)=\log a-\log b$	3. $\log a^b=b\log a$

Examples) Find the derivative dy/dx of the following expressions:

13) $y=\ln\sqrt{x^2+2x-3}$

Hard way:

$$y'=\frac{1}{\left(x^2+2x-3\right)^{1/2}}\left(\frac{1}{2}\right)\left(x^2+2x-3\right)^{-1/2}(2x+2)$$
$$y'=\frac{x+1}{x^2+2x-3}$$

Easy way:

$$y=\frac{1}{2}\ln\left(x^2+2x-3\right)$$
$$y'=\frac{2x+2}{2\left(x^2+2x-3\right)}$$
$$y'=\frac{x+1}{\left(x^2+2x-3\right)}$$

14) $y=\ln\dfrac{x^2}{3x-2}$

$$y=\ln\left(x^2\right)-\ln(3x-2)$$
$$y=2\ln x-\ln(3x-2)$$
$$y'=\frac{2}{x}-\frac{3}{3x-2}$$

15)
$$y=\ln\left(x\sqrt{3x-1}\right)$$
$$y=\ln x+\frac{1}{2}\ln(3x-1)$$
$$y'=\frac{1}{x}+\frac{3}{2(3x-1)}$$

16)
$$y=\ln\sqrt{\frac{x^2+1}{x^2-1}}$$
$$y=\frac{1}{2}\left[\ln\left(x^2+1\right)-\ln\left(x^2-1\right)\right]$$
$$y'=\frac{1}{2}\left[\frac{2x}{x^2+1}-\frac{2x}{x^2-1}\right]$$
$$y'=\frac{x}{x^2+1}-\frac{x}{x^2-1}$$

17)
$$y=\ln\frac{x\left(x^2+3\right)^3}{\sqrt[3]{2x^2+4}}$$
$$y=\ln x+3\ln\left(x^2+3\right)-\frac{1}{3}\ln\left(2x^2+4\right)$$
$$y'=\frac{1}{x}+\frac{6x}{x^2+3}-\frac{4x}{3\left(2x^2+4\right)}$$

18)
$$x^2-3\ln y+y^2=25$$
$$2x-\frac{3}{y}y'+2y\cdot y'=0$$
$$2xy-3y'+2y^2\cdot y'=0$$
$$2xy=y'\left(3-2y^2\right)$$
$$y'=\frac{2xy}{3-2y^2}$$

19) Find relative extrema of

$$y=\ln\left(x^2+4x+6\right)$$
$$y'=\frac{2x+4}{x^2+4x+6}=0$$

$x=-2$, rel min by 1st der. test

$(-2,\ln 2)$ is a rel. min

20) Find the equation of the tangent line to $y=4x^2-\ln x$ at $(1,4)$

$$y'=8x-\frac{1}{x},\ y'(1)=7$$
$$y-4=7(x-1)$$
$$y=7x-3$$

A technique not included on the AP exam but helpful to taking hard derivatives is called logarithmic differentiation. It essentially says to take the natural log of the hard expression in order to take advantage of the log rules. Example 21)

$$y=\frac{(2x-1)^3}{\sqrt{x^2+x+1}}\Rightarrow\ln y=3\ln(2x-1)-\frac{1}{2}\ln\left(x^2+x+1\right)$$
$$\frac{1}{y}\frac{dy}{dx}=\frac{6}{2x-1}-\frac{2x+1}{2\left(x^2+x+1\right)}\Rightarrow\frac{dy}{dx}=\left(\frac{6}{2x-1}-\frac{2x+1}{2\left(x^2+x+1\right)}\right)y=\left(\frac{6}{2x-1}-\frac{2x+1}{2\left(x^2+x+1\right)}\right)\frac{(2x-1)^3}{\sqrt{x^2+x+1}}$$

Differentiation of the Natural Log Function - Homework

Find the derivative dy/dx of the following expressions:

1. $y = \ln x^6$
$y' = \dfrac{6}{x}$

2. $y = \ln\left(x^2 - 5x - 2\right)$
$y' = \dfrac{2x-5}{x^2-5x-2}$

3. $y = (\ln x)^7$
$y' = \dfrac{7(\ln x)^6}{x}$

4. $y = x^3 \ln x$
$y' = x^2 + 3x^2 \ln x$

5. $y = \dfrac{x^4}{\ln x}$
$y' = \dfrac{4x^3 \ln x - x^3}{(\ln x)^2}$

6. $y = \dfrac{\ln x}{x^4}$
$y' = \dfrac{x^3 - 4x^3 \ln x}{x^8}$
$y' = \dfrac{1 - 4\ln x}{x^5}$

7. $y = x^2 - 4\ln x - 2$
$y' = 2x - \dfrac{4}{x}$

8. $y = \ln\left(\ln 3x^2\right)$
$y' = \left(\dfrac{1}{\ln 3x^2}\right)\left(\dfrac{6x}{3x^2}\right)$
$y' = \dfrac{2}{x \ln 3x^2}$

9. $y = \ln \sin x$
$y' = \left(\dfrac{1}{\sin x}\right)\cos x$
$y' = \cot x$

10. $y = \sin(\ln x)$
$y' = \dfrac{\cos(\ln x)}{x}$

11. $y = x \ln \tan x$
$y' = x\dfrac{\sec^2 x}{\tan x} + \ln \tan x$

12. $y = \dfrac{\tan x}{\ln x}$
$y' = \dfrac{\ln x \sec^2 x - \dfrac{\tan x}{x}}{(\ln x)^2}$
$y' = \dfrac{x \ln x \sec^2 x - \tan x}{x(\ln x)^2}$

13. $y = \ln\left[\left(3x^2 - 3x + 2\right)(5x-1)\right]$
$y = \ln\left(3x^2 - 3x + 2\right) + \ln(5x-1)$
$y' = \dfrac{6x-3}{3x^2-3x+2} + \dfrac{5}{5x-1}$

14. $y = \ln\dfrac{3x^2 - 3x + 2}{5x-1}$
$y = \ln\left(3x^2 - 3x + 2\right) - \ln(5x-1)$
$y' = \dfrac{6x-3}{3x^2-3x+2} - \dfrac{5}{5x-1}$

15. $y = \ln\sqrt{x^2 - 4x - 7}$
$y = \dfrac{1}{2}\ln\left(x^2 - 4x - 7\right)$
$y' = \dfrac{x-2}{x^2-4x-7}$

16. $y = \ln\sqrt[5]{3x^3 - 2x^2 + 5x - 1}$
$y = \dfrac{1}{5}\ln\left(3x^3 - 2x^2 + 5x - 1\right)$
$y' = \dfrac{9x^2 - 4x + 5}{5\left(3x^3 - 2x^2 + 5x - 1\right)}$

17. $y = \ln\sqrt{\dfrac{2x-1}{2x+1}}$
$y = \dfrac{1}{2}\left[\ln(2x-1) - \ln(2x+1)\right]$
$y' = \dfrac{1}{2}\left[\dfrac{2}{2x-1} - \dfrac{2}{2x+1}\right]$

18. $y = \ln\left[x \cdot \sqrt[3]{\dfrac{2x-1}{2x+1}}\right]$
$y = \ln x + \dfrac{1}{3}\left[\ln(2x-1) - \ln(2x+1)\right]$
$y' = \dfrac{1}{x} + \dfrac{1}{3}\left[\dfrac{2}{2x-1} - \dfrac{2}{2x+1}\right]$

Use implicit differentiation to find dy/dx.

19. $4x^2 + 2\ln y + y^3 = 12$
$8x + \frac{2}{y}\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$
$8xy + 2\frac{dy}{dx} + 3y^3\frac{dy}{dx} = 0$
$\frac{dy}{dx} = \frac{-8xy}{2+3y^3}$

20. $\ln xy - 3x = 4$
$\ln x + \ln y - 3x = 4$
$\frac{1}{x} + \frac{1}{y}\frac{dy}{dx} - 3 = 0$
$y + x\frac{dy}{dx} - 3xy = 0$
$\frac{dy}{dx} = \frac{3xy - y}{x}$

21. $\frac{\ln y}{x} = x + y$
$\ln y = x^2 + xy$
$\frac{1}{y}\frac{dy}{dx} = 2x + x\frac{dy}{dx} + y$
$\frac{dy}{dx} = 2xy + xy\frac{dy}{dx} + y^2$
$\frac{dy}{dx} = \frac{2xy + y^2}{1 - xy}$

Find the equation of the line tangent to the graph of the following functions at the indicated point.

22. $y = -2x^2 + \ln x - 1$ at $(1,-3)$
$y' = -4x + \frac{1}{x}, y'(1) = -3$
$y + 3 = -3(x-1)$
$y = -3x$

23. $y = 6 - x^2 - \ln(2x+1)$ at $(0,6)$
$y' = -2x - \frac{2}{2x+1}, y'(0) = -2$
$y - 6 = -2x$
$y = -2x + 6$

24. $y = x^3 - \ln\frac{x}{e}$ at $(1,2)$
$y' = 3x^2 - \frac{1}{x}, y'(1) = 2$
$y - 2 = 2(x-1)$
$y = 2x$

Find any relative extrema to the following functions.

25. $y = \frac{x^2}{4} - \ln x$
$y' = \frac{x}{2} - \frac{1}{x} = \frac{x^2 - 2}{x}$
Rel min at $x = \sqrt{2}$
y not defined at $x = -\sqrt{2}$

26. $y = \ln x - x$
$y' = \frac{1}{x} - 1 = \frac{1-x}{x}$
Rel max at $x = 1$

27. $y = 2x^2\ln x$
$y' = 2x + 4x\ln x$
$y' = 2x(1 + 2\ln x)$
$1 + 2\ln x = 0$
$\ln x = \frac{-1}{2}, x = \frac{1}{\sqrt{e}}$
Rel min at $x = \frac{1}{\sqrt{e}}$

28. $y = \frac{\ln x}{2x}$
$y' = \frac{2 - 2\ln x}{4x^2}$
$2 = 2\ln x$
$x = e$
Rel max at $x = e$

Use logarithmic differentiation to find dy/dx.

29. $y = \sqrt{(x-3)(x-4)(x-5)}$
$\ln y = \frac{1}{2}\left[\ln(x-3) + \ln(x-4) + (x-5)\right]$
$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x-3} + \frac{1}{x-4} + \frac{1}{x-5}\right]$
$\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x-3} + \frac{1}{x-4} + \frac{1}{x-5}\right]\sqrt{(x-3)(x-4)(x-5)}$

30. $y = \sqrt{\frac{x^2-2}{x^2+2}}$
$\ln y = \frac{1}{2}\left[\ln(x^2-2) - \ln(x^2+2)\right]$
$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{2x}{x^2-2} - \frac{2x}{x^2+2}\right]$
$\frac{dy}{dx} = \left[\frac{x}{x^2-2} - \frac{x}{x^2+2}\right]\sqrt{\frac{x^2-2}{x^2+2}}$

the Natural Log Function and Integration - Classwork

The derivative rules which we just learned will now produce the following integration rules:

$\int \frac{1}{x}\,dx = \ln|x| + C$ and if u is a differentiable function of x, $\int \frac{1}{u}\,du = \ln|u| + C$

Examples) Find the following:

1) $\int \frac{4}{x}\,dx$
$4\ln|x| + C$

2) $\int \frac{1}{5x-2}\,dx$
$u = 5x-2, du = 5dx$
$\frac{1}{5}\int \frac{5}{5x-2}\,dx = \frac{1}{5}\int \frac{du}{u}$
$\frac{1}{5}\ln|5x-2| + C$

3) $\int \frac{4}{3-6x}\,dx$
$u = 3-6x, du = -6dx$
$-\frac{4}{6}\int \frac{-6}{3-6x}\,dx = -\frac{2}{3}\int \frac{du}{u}$
$\frac{-2}{3}\ln|3-6x| + C$

4) $\int \frac{7x}{x^2-4}\,dx$
$u = x^2-4, du = 2xdx$
$\frac{7}{2}\int \frac{2x}{x^2-4}\,dx = \frac{7}{2}\int \frac{du}{u}$
$\frac{7}{2}\ln|x^2-4| + C$

When you take integrals of fractions, you usually think u-substitution with the u being the denominator generating a ln function. But not always.

5) $\int \frac{x}{\sqrt{16-x^2}}\,dx$
$u = 16-x^2, du = -2xdx$
$\frac{1}{-2}\int (-2x)(16-x^2)^{-1/2}dx$
$-(16-x^2)^{1/2} + C$

6) $\int \frac{1}{\sqrt[3]{2x-1}}\,dx$
$u = 2x-1, du = 2xdx$
$\frac{1}{2}\int (2)(2x-1)^{-1/3}dx$
$\frac{3}{4}(2x-1)^{2/3} + C$

7) $\int \frac{(\ln x)^4}{x}\,dx$
$u = \ln x, du = 1/x\,dx$
$\int u^4 du$
$\frac{(\ln x)^5}{5} + C$

8) $\int \frac{x^2-2x+1}{x}\,dx$
$\int \left(x - 2 + \frac{1}{x}\right)dx$
$\frac{x^2}{2} - 2x + \ln|x| + C$

9) $\int \tan x\,dx = \int \frac{\sin x}{\cos x}dx$
$u = \cos x, du = -\sin xdx$
$-\int \frac{du}{u}dx$
$-\ln|\cos x| + C$

10) $\int \cot 3x\,dx = \int \frac{\cos 3x}{\sin 3x}dx$
$u = \sin 3x, du = 3\cos 3xdx$
$\frac{1}{3}\int \frac{du}{u}dx$
$\frac{1}{3}\ln|\sin 3x| + C$

11) $\int \frac{\cos x}{2+\sin x}\,dx$
$u = 2+\sin x, du = \cos xdx$
$\int \frac{du}{u}dx$
$\ln|2+\sin x| + C$

12) $\int \frac{1}{\cos^2 x \tan x}\,dx$
$u = \tan x, du = \sec^2 xdx$
$\int \frac{du}{u}dx$
$\ln|\tan| + C$

13) $\int_0^4 \frac{4}{2x+1}\,dx$
$u = 2x+1, du = 2dx$
$x = 0, u = 1, x = 4, u = 9$
$\frac{4}{2}\int \frac{du}{u} = 2\ln u\Big|_1^9$
$2\ln 9 = 2\ln 3^2 = 4\ln 3$

14) $\int_e^{e^2} \frac{1}{x\ln x}\,dx$
$u = \ln x, du = 1/x\,dx$
$x = e, u = 1, x = e^2, u = 2$
$\int \frac{du}{u} = \ln u\Big|_1^2$
$\ln 2$

15) $\int_0^{\pi/2} \frac{\sin x}{1+\cos x}\,dx$
$u = 1+\cos x, du = -\sin xdx$
$x = 0, u = 2, x = \pi/2, u = 1$
$\int \frac{du}{u} = \ln u\Big|_1^2$
$\ln 2$

the Natural Log Function and Integration - Homework

Find the following:

1. $\int \frac{2}{x+2}\,dx$

$u = x+2, du = dx$

$2\int \frac{1}{u}\,du = 2\ln|x+2| + C$

2. $\int \frac{5}{4-3x}\,dx$

$u = 4-3x, du = -3dx$

$\frac{-5}{3}\int \frac{du}{u} = \frac{-5}{3}\ln|4-3x| + C$

3. $\int \frac{x}{x^2-1}\,dx$

$u = x^2-1, du = 2xdx$

$\frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|x^2-1| + C$

4. $\int \frac{x^2}{5-x^3}\,dx$

$u = 5-x^3, du = -3x^2dx$

$-\frac{1}{3}\int \frac{du}{u} = \frac{-1}{3}\ln|5-x^3| + C$

5. $\int \frac{x}{\sqrt[3]{x^2+1}}\,dx$

$u = x^2+1, du = 2x\,dx$

$\frac{1}{2}\int u^{-1/3}du = \frac{3}{4}\left(x^2+1\right)^{2/3} + C$

6. $\int \frac{x^2-3x-5}{x}\,dx$

$\int\left(x-3-\frac{5}{x}\right)dx$

$\frac{x^2}{2} - 3x - 5\ln|x| + C$

7. $\int \frac{2x-5}{x^2-5x-1}\,dx$

$u = x^2-5x-1, du = (2x-5)\,dx$

$\int \frac{du}{u} = \ln|x^2-5x-1| + C$

8. $\int \frac{x+4}{x^2+8x-3}\,dx$

$u = x^2+8x-3, du = (2x+8)\,dx$

$\frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|x^2+8x-3| + C$

9. $\int \frac{(\ln x)^5}{x}\,dx$

$u = \ln x, du = \frac{1}{x}\,dx$

$\int u^5 du = \frac{(\ln x)^6}{6} + C$

10. $\int \frac{5}{x\ln x}\,dx$

$u = \ln x, du = \frac{1}{x}\,dx$

$5\int \frac{du}{u} = 5\ln(\ln(x)) + C$

11. $\int \tan 4\theta\,d\theta$

$\int \frac{\sin 4\theta}{\cos 4\theta}\,d\theta$

$u = \cos 4\theta, du = -4\sin 4\theta$

$\frac{1}{4}\int \frac{du}{u}\,du = \frac{-\ln|\cos(4\theta)|}{4} + C$

12. $\int \frac{\sin\theta}{4-3\cos\theta}\,d\theta$

$u = 4-3\cos\theta, du = 3\sin\theta$

$\frac{1}{3}\int \frac{du}{u}\,du = \frac{\ln|4-3\cos\theta|}{3} + C$

13. $\int_0^4 \frac{4}{2x+1}\,dx$

$u = 2x+1, du = 2dx$

$x = 0, u = 1 \quad x = 4, u = 9$

$\frac{4}{2}\int_1^9 \frac{du}{u} = 2(\ln u)\Big|_1^9$

$2(\ln 9 - \ln 1) = 2\ln 3^2 = 4\ln 3$

14. $\int_1^e \frac{(1+\ln x)^3}{x}\,dx$

$u = 1+\ln x, du = \frac{dx}{x}$

$x = 0, u = 1 \quad x = 3, u = 2$

$\int_1^2 u^3du = \left[\frac{u^4}{4}\right]_1^2 = 4 - \frac{1}{4} = \frac{15}{4}$

15. $\int_0^{\pi/2} \frac{1-\sin x}{x+\cos x}\,dx$

$u = x+\cos x, du = 1-\sin x$

$x = 0, u = 1 \quad x = \pi/2, u = \pi/2$

$\int_1^{\pi/2} \frac{du}{u} = (\ln u)\Big|_1^{\pi/2} = \ln\left(\frac{\pi}{2}\right)$

Derivatives and Integrals of Expressions with "e" - Classwork

Let us try to take the derivative of $y = e^x$. Again, it seems as if that there is no rule (power, product, quotient, trig, ln) to take it. Let's examine this by use of the calculator. Set Y1 = e^x and Y2 = NDeriv(e^x,*x*,*x*). Then set a table to look at these values.

x	-4	-3	-2	-1	0	1	2	3	4
e^x	0.018	0.050	0.135	0.368	1.000	2.718	7.389	20.086	54.598
$\frac{d}{dx}\left(e^x\right)$	0.018	0.050	0.135	0.368	1.000	2.718	7.389	20.086	54.598

It should be obvious (and surprising) what is happening. Let's try and prove it. Let's take the derivative of $y = e^x$ using logarithmic differentiation.

$$\ln y = x \ln e = x$$
$$\frac{1}{y}\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = y = e^x$$

So we end up with the result that equations in the form of $y = Ce^x$ are the only equations (with the exception of $y = 0$) whose derivative is the same as the expression itself.

$$\frac{d}{dx}\left[e^x\right] = e^x \text{ and if } u \text{ is a differentiable function of } x \text{ then } \frac{d}{dx}\left[e^u\right] = e^u \frac{du}{dx}$$

Examples) Find the derivative dy/dx of the following expressions:

1) $y = e^{5x}$; $y' = 5e^{5x}$

2) $y = 4e^{1-2x}$; $y' = -8e^{1-2x}$

3) $y = e^{x^2-3x-1}$; $y' = (2x-3)e^{x^2-3x-1}$

4) $y = 2e^{\sqrt{x}}$; $y' = \dfrac{e^{\sqrt{x}}}{\sqrt{x}}$

We now have 6 basic rules for derivatives: power, product, quotient, trig (6), ln, and now *e*.

5) $y = \left(e^x + 3\right)^2$; $y' = 2e^x\left(e^x + 3\right)$

6) $y = xe^x$; $y' = xe^x + e^x$

7) $y = \sin\left(e^x\right)$; $y' = e^x \cos\left(e^x\right)$

8) $y = e^{\sin x}$; $y' = e^{\sin x}(\cos x)$

9) $y = \ln\left(x + e^x\right)$; $y' = \dfrac{1 + e^x}{x + e^x}$

10) $y = \dfrac{e^{4x}}{x}$; $y' = \dfrac{4xe^{4x} - e^{4x}}{x^2}$

11) $y = \dfrac{x}{e^{4x}}$; $y' = \dfrac{e^{4x} - 4xe^{4x}}{e^{8x}}$; $y' = \dfrac{1 - 4x}{e^{4x}}$

12) $y = \sqrt[3]{e^x}$; $y' = \dfrac{1}{3}e^{x/3}$

Find dy/dx by implicit differentiation:

13)
$$xe^y + 8x - 3y = 0$$
$$xe^y \frac{dy}{dx} + e^y + 8 - 3\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{e^y + 8}{3 - xe^y}$$

Find the second derivative of the function:

14)
$$y = e^x - e^{-x}$$
$$y' = e^x + e^{-x}$$
$$y'' = e^x - e^{-x}$$

15. Find relative extrema and inflection point(s) for the function $y = e^x - e^{-x}$.

$y = e^x - e^{-x}$

$y' = e^x + e^{-x}$ Never equals zero, always positive (always increasing)

$y'' = e^x - e^{-x}$ Equals zero when $x = 0$, concave down if $x < 0$, concave up if $x > 0$

16. Find the area of the largest rectangle that can be inscribed under the curve $y = 3 - e^x$ in the first quadrant.

$A = x\left(3 - e^x\right) = 3x - xe^x$

$A' = 3 - xe^x - e^x = 0$

$x = .618$ (Graph and find zeros)

$A = .618\left(3 - e^{.618}\right) = .707$

Obviously, since the derivative of $y = e^x$ is e^x, it follows that the integral formula should be as simple.

$\int e^x dx = e^x + C$ and if u is a differentiable function of x then $\int e^u du = e^u + C$

Examples) Find the following:

17) $\int e^{5x} dx$

$u = 5x, du = 5dx$

$\frac{1}{5}\int e^u du = \frac{1}{5}e^{5x} + C$

18) $\int 4e^{1-x} dx$

$u = 1 - x, du = -dx$

$-4\int e^u du = -4e^{1-x} + C$

19) $\int 2xe^{x^2} dx$

$u = x^2, du = 2dx$

$\int e^u du = e^{x^2} + C$

20) $\int \frac{e^{1/x}}{2x^2} dx$

$u = \frac{1}{x}, du = \frac{-1}{x^2} dx$

$-\frac{1}{2}\int e^u du = \frac{-e^{1/x}}{2} + C$

21) $\int \cos x \cdot e^{\sin x} dx$

$u = \sin x, du = \cos x dx$

$\int e^u du = e^{\sin x} + C$

22) $\int \frac{e^x}{4 - e^x} dx$

$u = 4 - e^x, du = -e^x dx$

$-\int \frac{du}{u} = -\ln\left|4 - e^x\right| + C$

23) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

$u = e^x + e^{-x}, du = \left(e^x - e^{-x}\right)dx$

$\int \frac{du}{u} = \ln\left|e^x + e^{-x}\right| + C$

24) $\int \frac{e^{4x} + 2e^x + 1}{e^x} dx$

$\int \left(e^{3x} + 2 + e^{-x}\right) dx$

$\frac{1}{3}e^{3x} + 2x - e^{-x} + C$

Find the area bounded by the curves and lines.Verify by calculator.

19) $y = e^{-x}, y = 0, x = 0, x = 1$

$A = \int_0^1 e^{-x} dx$

$A = -e^{-x}\Big|_0^1$

$A = 1 - \frac{1}{e}$

20) $y = \frac{e^x}{2 + e^x}, x = 1, y = 0$

$A = \int_0^1 \frac{e^x}{2 + e^x} dx$

$A = \ln\left(2 + e^x\right)\Big|_0^1$

$A = \ln(2 + e) - \ln 3 = \ln\left(\frac{2 + e}{3}\right)$

21) $y = e^x \sin e^x, x = 0, y = 0$

$A = \int_0^{1.145} e^x \sin e^x dx$

$A = -\cos e^x\Big|_0^{1.145}$

$A = -\cos e^{1.145} + \cos 1$

22) Find the volume when the first quadrant region R bounded by $y = e^{x/2}$ and $x = 2$ is rotated about the x-axis.

$y = e^{x/2}$

$V = \pi \int_0^2 \left(e^{x/2}\right)^2 dx$

$V = \pi e^x\Big|_0^2$

$V = \pi\left(e^2 - 1\right)$

Finally, occasionally, we have to take derivatives of exponential functions with bases other than e. Using the fact that $a^x = e^{(\ln a)x}$, we can take the derivative by saying that $\frac{d}{dx} a^x = e^{(\ln a)x} \cdot \ln a = a^x \cdot \ln a$. You need to know that:

$$\frac{d}{dx} a^x = a^x \cdot \ln a \qquad \text{and} \qquad \frac{d}{dx} a^u = a^u \cdot \ln a \cdot \frac{du}{dx}$$

Examples) Find the derivatives of:

23) $y = 5^x$

$y' = 5^x \ln 5$

24) $y = 3^{x^2 - 2x}$

$y' = \left(3^{x^2 - 2x}\right)(\ln 3)(2x - 2)$

25) $y = x6^{-x}$

$y' = \left(x6^{-x}\right)(\ln 6)(-1) + 6^{-x}$

Derivatives and Integrals of Expressions with "e" - Homework

Find the derivatives of the following functions:

1. $y = e^{4x}$
$y' = 4e^{4x}$

2. $y = 16e^{-2x}$
$y' = -32e^{-2x}$

3. $y = x^3 e^x$
$y' = x^3 e^x + e^x\left(3x^2\right)$

4. $y = \dfrac{e^x}{x^6}$
$y' = \dfrac{x^6 e^x - 6x^5 e^x}{x^{12}}$
$y' = \dfrac{e^x(x-6)}{x^7}$

5. $y = e^x \tan x$
$y' = e^x \sec^2 x + e^x \tan x$

6. $y = 2\cos e^x$
$y' = -2e^x \sin e^x$

7. $y = \dfrac{e^x}{\ln x}$
$y' = \dfrac{(\ln x)(e^x) - e^x(1/x)}{(\ln x)^2}$
$y' = \dfrac{x(\ln x)(e^x) - e^x}{x(\ln x)^2}$

8. $y = \dfrac{\ln x}{e^x}$
$y' = \dfrac{e^x(1/x) - (\ln x)(e^x)}{(e^x)^2}$
$y' = \dfrac{e^x - x(\ln x)(e^x)}{xe^{2x}}$
$y' = \dfrac{1 - x(\ln x)}{xe^x}$

9. $y = \left(e^x - 2x - 1\right)^3$
$y' = 3\left(e^x - 2x - 1\right)^2\left(e^x - 2\right)$

10. $y = \sqrt{e^{3x} - 4x}$
$y' = \dfrac{3e^{3x} - 4}{2\left(e^{3x} - 4x\right)^{1/2}}$

11. $y = -4e^{\sec x}$
$y' = -4e^{\sec x}(\sec x \tan x)$

12. $y = \dfrac{3}{e^x + e^{-x}}$
$y' = \dfrac{-3\left(e^x - e^{-x}\right)}{\left(e^x + e^{-x}\right)^2}$

13. $y = \ln\left(\dfrac{1 - e^x}{1 + e^x}\right)$
$y = \ln\left(1 - e^x\right) - \ln\left(1 + e^x\right)$
$y' = \dfrac{-e^x}{1 - e^x} - \dfrac{e^x}{1 + e^x}$

14. $y = e^x(\sin x - \cos x)$
$y' = e^x(\cos x + \sin x) +$
$e^x(\sin x - \cos x)$
$y' = 2e^x \sin x$

15. $y = \pi^x$
$y' = \pi^x \ln \pi$

16. $y = 10^{x^2 - \sin x}$
$y' = \left(10^{x^2 - \sin x}\right)(\ln 10)(2x - \cos x)$

Use implicit differentiation to find dy/dx

17. $xe^y - 2x - 3y = 0$
$xe^y \dfrac{dy}{dx} + e^y - 2 - 3\dfrac{dy}{dx} = 0$
$\dfrac{dy}{dx} = \dfrac{e^y - 2}{3 - xe^y}$

Find the second derivative of the following

18. $y = \sqrt{x} - e^x \ln x$
$y' = \dfrac{1}{2}x^{-1/2} - \dfrac{e^x}{x} - e^x \ln x$
$y'' = -\dfrac{1}{4x^{3/2}} - \dfrac{xe^x - e^x}{x^2} - \dfrac{e^x}{x} - e^x \ln x$
$y'' = -\dfrac{1}{4x^{3/2}} - \dfrac{2xe^x - e^x}{x^2} - e^x \ln x$

Find the extrema for the following functions:

19. $y = e^{\left(\frac{-x^2}{2}\right)}$
$y' = -xe^{\left(\frac{-x^2}{2}\right)}$
Rel. max at $(0,1)$

20. $y = 2xe^{-x}$
$y' = -2xe^{-x} + 2e^{-x}$
$y' = -2e^{-x}(x-1)$
Rel. max at $\left(1, \frac{2}{e}\right)$

21. $y = x^2e^{-x}$
$y' = -x^2e^{-x} + 2xe^{-x}$
$y' = -xe^{-x}(x-2)$
Rel. max at $\left(2, \frac{4}{e^2}\right)$, rel. min at $(0,0)$

Find the following integrals

22. $\int e^{6x} dx$

$u = 6x, du = 6\ dx$
$\frac{1}{6}\int e^u du$
$\frac{1}{6}e^{6x} + C$

23. $\int 4e^{-2x} dx$

$u = -2x, du = -2\ dx$
$4\left(\frac{-1}{2}\right)\int e^u du$
$-2e^{-2x} + C$

24. $\int \sin x \cdot e^{\cos x}\ dx$

$u = \cos x, du = -\sin\ dx$
$-\int e^u du$
$-e^{\cos x} + C$

25. $\int \frac{e^{\tan x}}{\cos^2 x}\ dx$

$u = \tan x, du = \sec^2 x\ dx$
$\int e^u du$
$e^{\tan x} + C$

26. $\int \left(e^x + e^{-x}\right)^2\ dx$

$\int \left(e^{2x} + 2 + e^{2x}\right)$
$\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} + C$

27. $\int e^x\sqrt{4 - e^x}\ dx$

$u = 4 - e^x, du = -e^x\ dx$
$-\int u^{1/2} du = -\frac{2u^{3/2}}{3}$
$\frac{-2\left(4 - e^x\right)^{3/2}}{3} + C$

28. $\int \frac{e^x + e^{-x}}{e^x - e^{-x}}\ dx$

$u = e^x - e^{-x}, du = \left(e^x + e^{-x}\right)\ dx$
$\int \frac{du}{u} du = -\ln|u|$
$\ln\left|e^x - e^{-x}\right| + C$

29. $\int \frac{e^{2x} + 2e^{-x} + 1}{e^x}\ dx$

$\int \left(e^x + 2e^{-2x} + e^{-x}\right)\ dx$
$e^x - e^{-2x} - e^{-x} + C$

30. $\int \frac{e^{-x}}{4 + e^{-x}}\ dx$

$u = 4 + e^{-x}, du = -e^{-x}\ dx$
$-\int \frac{du}{u} = \ln|u|$
$-\ln\left(4 + e^{-x}\right) + C$

31. $\int e^{-x}\tan\left(e^{-x}\right)\ dx$

$u = \cos e^{-x}, du = e^{-x}\left(\sin e^{-x}\right)\ dx$
$\int \frac{du}{u} du = -\ln|u|$
$\ln\left|\cos e^{-x}\right| + C$

32. $\int_1^9 \frac{e^{\sqrt{x}}}{2\sqrt{x}}\ dx$

$u = \sqrt{x}, du = \frac{1}{2\sqrt{x}}\ dx$
$x = 1, u = 1 \quad x = 9, u = 3$
$\int_1^3 e^u du = e^u\Big|_1^3 = e^3 - e$

33. $\int_1^{e^3} \frac{1}{x}\ dx$

$\ln x\Big|_1^{e^3} = \ln e^3 - \ln 1$
3

Find the area of the region bounded by the graphs of the functions:

34. $y = e^{-x}, x = 0, y = 0, x = 4$

$A = \int_0^4 e^{-x}\,dx$

$1 - \frac{4}{e}$

35. $y = xe^{-x^2/2}, x = 0, y = 0, x = \sqrt{2}$

$A = \int_0^{\sqrt{2}} xe^{-x^2/2}\,dx$

$1 - \frac{1}{e}$

36. Let R be the region bounded by $y = 6e^{-0.2x}$ and $y = \sqrt{x}$ and the lines $x = 1$ and $x = 4$. Find the volume when R is rotated about the x-axis.

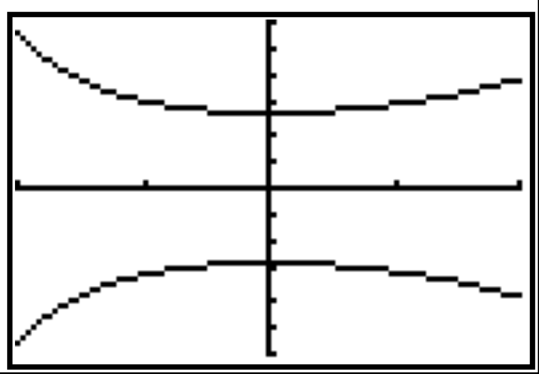
$V = \pi\int_1^4 \left[36e^{-0.4x} - x\right]dx$

$V = 34.658\pi$

37. Find the volume when the area between the graphs of the following functions is rotated about the x-axis: $y = e^{x/2} + e^{-x/2}, y = 0, x = -2, x = 2$

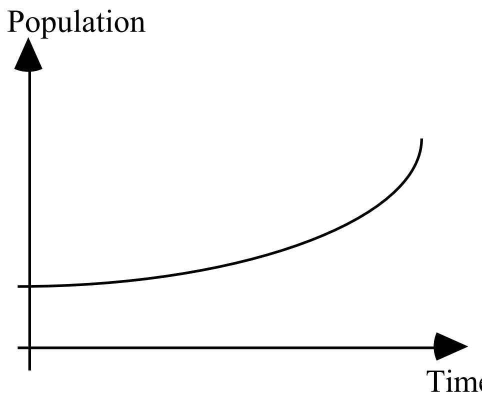
$V = \pi\int_{-2}^{2} \left(e^{x/2} + e^{-x/2}\right)^2 dx$

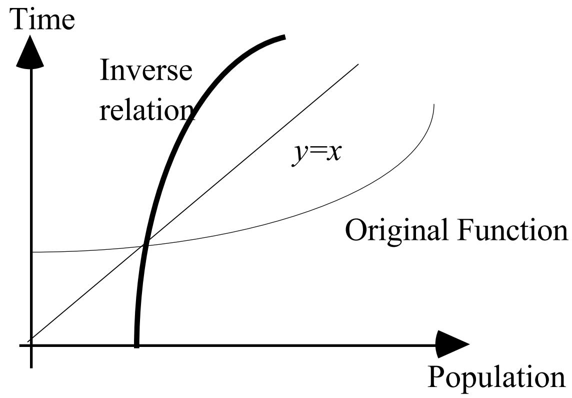
$V = 22.507\pi$



Inverse Trig Functions - Classwork

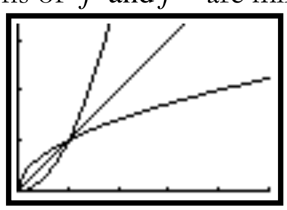
The left hand graph below shows how the population of a certain city may grow as a function of time. If you are interested in finding the time at which the population reaches a certain value, it may be more convenient to reverse the variables and write time as a function of population. The relation you get by interchanging the two variables is called the **inverse** of the original function. The graph of the inverse is shown on the right graph below.



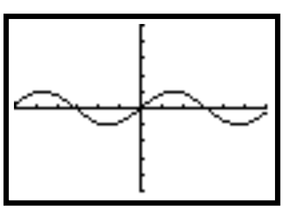


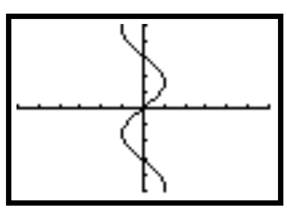
For a linear function such as $y = 2x + 6$, interchanging the variables gives $x = 2y + 6$ for the inverse relation. Solving for *y* in terms of *x* gives $y = .5x - 3$. The symbol f^{-1}, pronounced "f inverse," is used for the inverse function. If $f(x) = 2x + 6$, then $f^{-1}(x) = 0.5x - 3$.

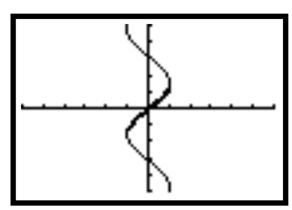
If f^{-1} turns out to be a function (passes the vertical line test), the the original function f is said to be **invertible**. Remember that the -1 exponent does *not* mean the reciprocal of $f(x)$. The inverse of a function undoes what the function did to *x*. That is $f^{-1}(f(x)) = x$. If $f(x) = x^2$, then $f^{-1}(x) = \sqrt{x}$. Note that if the same scales are used for the two axes, then the graphs of f and f^{-1} are mirror images with respect to the $45°$ line $y = x$.



The inverses of the trigonometric functions follow from the definition. For instance, if the function is $y = \sin x$, the inverse function is given by $x = \sin y$. When we solve for *y*, we get $y = \sin^{-1} x$. The symbol **arcsin x** is sometimes used to help you distinguish $\sin^{-1} x$ from $1/\sin x$. Here are the graphs of $y = \sin x$ and $y = \sin^{-1} x$.

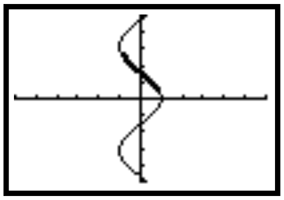


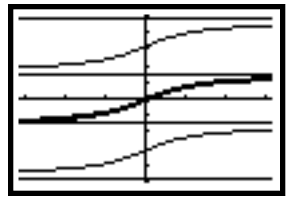




It is obvious that the inverse sine relation is not a function. There are many values of *y* for the same value of *x*. To create a function that is the inverse of sin *x* it is customary to restrict the range to $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$. This includes only the branch of the graph nearest the origin. (3rd picture above).

Below are pictures of the inverse cosine function and inverse tangent function. Note that in order to ensure that these relations are functions, we have to restrict the range.





So, we have these definitions:

$y = \sin^{-1} x$ if and only if $\sin y = x$ and $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$y = \cos^{-1} x$ if and only if $\cos y = x$ and $y \in [0, \pi]$

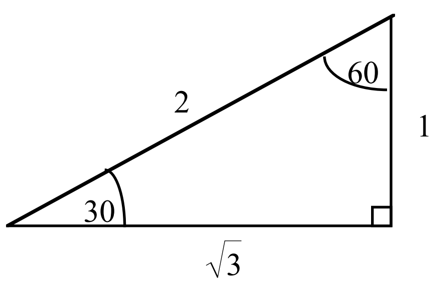
$y = \tan^{-1} x$ if and only if $\tan y = x$ and $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

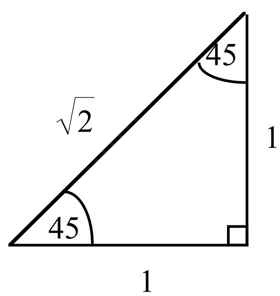
You must know conversions of degrees to radians and special triangles.

2π radians $= 360°$ or π radians $= 180°$. Some of the relationships that you should know are:

Degrees	30	45	60	90	120	135	150	180
Radians	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π

In a $30° - 60° - 90°$ triangle, the sides are always in the proportion $1 - \sqrt{3} - 2$
In a $45° - 45° - 90°$ triangle, the sides are always in the proportion $1 - 1 - \sqrt{2}$





Example 1) Evaluate each of the following:

a) $\arcsin\left(-\frac{1}{2}\right)$ → $-\frac{\pi}{6}$

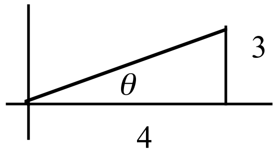
b. $\cos^{-1} 0$ → $\frac{\pi}{2}$

c. $\tan^{-1}\sqrt{3}$ → $\frac{\pi}{3}$

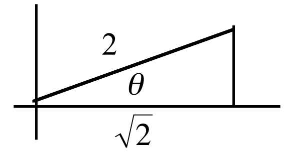
d. $\csc^{-1}\left(-\sqrt{2}\right)$ → $-\frac{\pi}{4}$

Example 2) Evaluate the following. Make a picture to describe the situation.

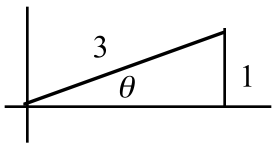
a. $\sin\left(\arctan\frac{3}{4}\right)=\frac{3}{5}$



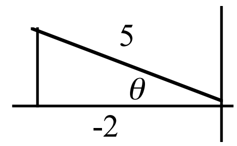
b. $\tan\left(\arccos\frac{\sqrt{2}}{2}\right)=1$



c. $\sec\left(\sin^{-1}\frac{1}{3}\right)=\frac{3}{\sqrt{8}}$

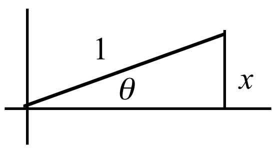


d. $\cot\left(\cos^{-1}\frac{-2}{5}\right)=\frac{-2}{\sqrt{21}}$

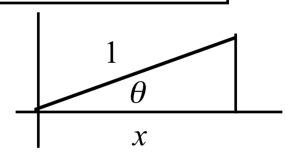


Example 3) Evaluate the following. Make a picture to describe the situation.

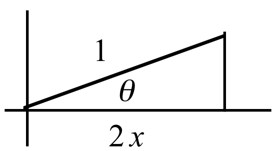
a. $\cos\left(\sin^{-1}x\right)=\sqrt{1-x^2}$



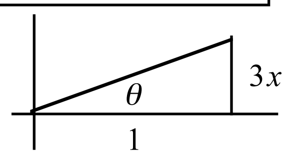
b. $\tan\left(\cos^{-1}x\right)=\frac{\sqrt{1-x^2}}{x}$



c. $\sin\left(\cos^{-1}2x\right)=\sqrt{1-4x^2}$



d. $\sin\left(\tan^{-1}3x\right)=\frac{3x}{\sqrt{1+9x^2}}$



So now we can take derivatives of inverse trig functions. Find $\frac{d}{dx}\left(\sin^{-1}x\right)$

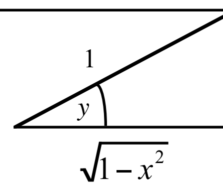
.

$y=\sin^{-1}x \Leftarrow \sin y = x$

Draw a picture

the angle is y, opposite $=x$, hypotenuse $=1$

Remaining side is $\sqrt{1-x^2}$



Since $\sin y = x$, take the derivative of each side

$$\cos y\frac{dy}{dx}=1 \quad \text{or} \quad \frac{dy}{dx}=\frac{1}{\cos y} \quad \text{or} \quad \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}$$

Eaxmple 4) Take the derivative of

a. $y=\cos^{-1}x$, $\frac{dy}{dx}=\frac{-1}{\sqrt{1-x^2}}$

b. $y=\tan^{-1}x$, $\frac{dy}{dx}=\frac{1}{1+x^2}$

The derivatives of the three inverse trig functions are as follows:

$$\frac{d}{dx}\left(\sin^{-1} u\right) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$
$$\frac{d}{dx}\left(\cos^{-1} u\right) = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$$
$$\frac{d}{dx}\left(\tan^{-1} u\right) = \frac{1}{1+u^2}\frac{du}{dx}$$

Example 5) Find the derivatives of

a. $y = \sin^{-1} 4x$

$$\frac{dy}{dx} = \frac{4}{\sqrt{1-16x^2}}$$

b. $y = \tan^{-1} x^3$

$$\frac{dy}{dx} = \frac{3x^2}{1+x^6}$$

c. $y = \left(\cos^{-1}\frac{x}{2}\right)^3$

$$\frac{dy}{dx} = 3\left(\cos^{-1}\frac{x}{2}\right)^2 \frac{-1}{\sqrt{1-\frac{x^2}{4}}}\left(\frac{1}{2}\right) = \frac{-3\left(\cos^{-1}\frac{x}{2}\right)^2}{\sqrt{4-x^2}}$$

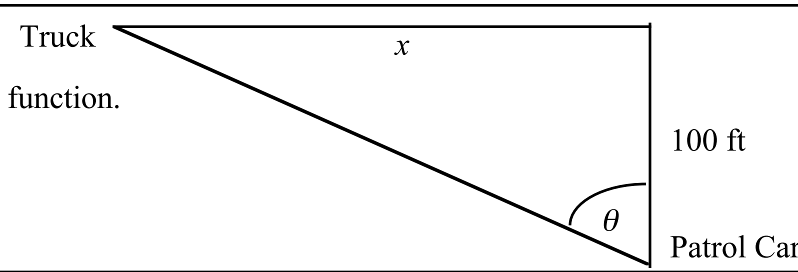
d. $y = x\sin^{-1} x + \sqrt{1-x^2}$

$$\frac{dy}{dx} = \frac{x}{\sqrt{1-x^2}} + \sin^{-1} x + \frac{-2x}{2\sqrt{1-x^2}} = \sin^{-1} x$$

Example 6) An officer in a patrol car sitting 100 feet from the highway observes a truck approaching. At a particular instant *t* seconds, the truck is *x* feet down the road. The line of sight to the truck makes an angle of θ radians to a perpendicular line to the road.

a. Express θ as an inverse trig function.

$$\theta = \tan^{-1}\frac{x}{100}$$



b. Find $\dfrac{d\theta}{dt}$

c. When the truck is at $x = 500$ ft, the angle is observed to be changing at a rate $\dfrac{d\theta}{dt}$ = -2 degrees/sec. How fast is the car going in ft/sec and mph?

$$\frac{d\theta}{dt} = \frac{100}{10000+x^2}\frac{dx}{dt}$$
$$\frac{-2\pi}{180} = \frac{100}{260000}\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = -90.757 \text{ ft/sec} = 61.880 \text{ mph}$$

Inverse Trig Functions - Homework

1. Evaluate each of the following:

a. $\arccos\frac{1}{2}$ — $\frac{\pi}{3}$

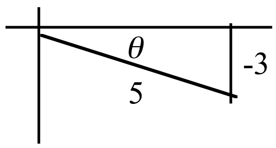
b. $\cot^{-1}\sqrt{3}$ — $\frac{\pi}{6}$

c. $\sin^{-1}\frac{-\sqrt{3}}{2}$ — $-\frac{\pi}{3}$

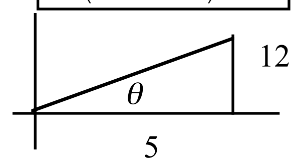
d. $\sec^{-1}-\frac{2\sqrt{3}}{3}$ — $\frac{5\pi}{6}$

2. Evaluate the following. Make a picture to describe the situation.

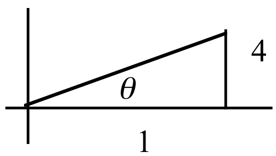
a. $\cos\left(\arcsin\frac{-3}{5}\right)=\frac{4}{5}$



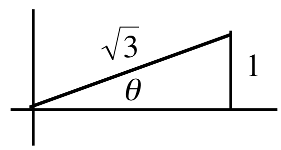
b. $\sin\left(\arctan\frac{12}{5}\right)=\frac{12}{13}$



c. $\csc\left(\cot^{-1}4\right)=\sqrt{17}$

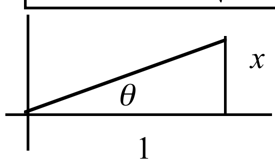


d. $\tan\left(\csc^{-1}\sqrt{3}\right)=\frac{\sqrt{2}}{2}$

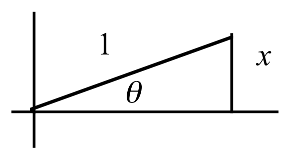


3. Evaluate the following. Make a picture to describe the situation.

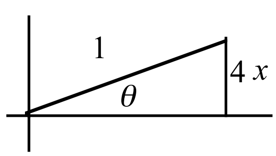
a. $\cos\left(\tan^{-1}x\right)=\frac{1}{\sqrt{1+x^2}}$



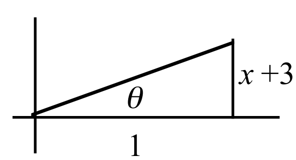
b. $\sec\left(\sin^{-1}x\right)=\frac{1}{\sqrt{1-x^2}}$



c. $\tan\left(\sin^{-1}4x\right)=\frac{4x}{\sqrt{1-16x^2}}$



d. $\cos\left(\tan^{-1}(x+3)\right)=\frac{1}{\sqrt{x^2+6x+10}}$



4. Find the derivatives of

a. $y=\cos^{-1}(3x)$

$$\frac{dy}{dx}=\frac{-3}{\sqrt{1-9x^2}}$$

b. $y=\sin^{-1}\left(x^2-1\right)$

$$\frac{dy}{dx}=\frac{2x}{\sqrt{1-\left(x^2-1\right)^2}}=\frac{2x}{\sqrt{2x^2-x^4}}$$

c. $y = \left(\tan^{-1} 2x\right)^5$

$\dfrac{dy}{dx} = 5\left(\tan^{-1} 2x\right)^4 \dfrac{2}{1+4x^2} = \dfrac{10\left(\tan^{-1} 2x\right)^4}{1+4x^2}$

d. $y = \sqrt{\left(\cos^{-1} 10x\right)}$

$\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{\left(\cos^{-1} 10x\right)}} \dfrac{-10}{\sqrt{1-100x^2}}$

$\dfrac{dy}{dx} = \dfrac{-5}{\sqrt{\left(\cos^{-1} 10x\right)\left(1-100x^2\right)}}$

e. $y = \arctan\sqrt{x}$

$\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}(1+x)}$

f. $y = \sin\left(\cos^{-1} t\right)$

$\dfrac{dy}{dt} = \cos\left(\cos^{-1} t\right)\dfrac{-1}{\sqrt{1-t^2}}$

$\dfrac{dy}{dt} = \dfrac{-t}{\sqrt{1-t^2}}$

5. Find any relative extrema of $y = \arcsin x - x$

$y = \arcsin x - x \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}} - 1 = 0$

$x = 0$, Critical points $x = -1,1 \Rightarrow \dfrac{dy}{dx} > 0 \ (-1,0)\cup(0,1)$ No relative extrema

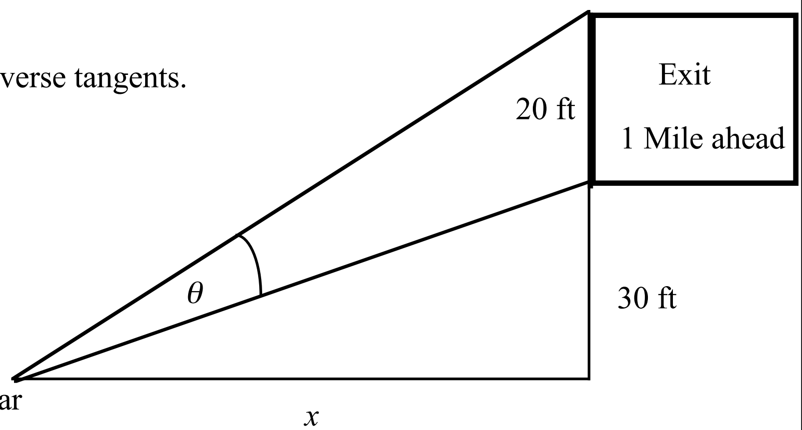
6. The base of a 20 foot tall exit sign is 30 feet above the driver's eye level. When cars are far away, the sign is hard to read because of the distance. When they are close, the sign is hard to read because the driver has to look up at a steep angle. The sign is easiest to read when the distance x is such that the angle θ at the driver's eye is as large as possible.

a) Write θ as the difference of 2 inverse tangents.

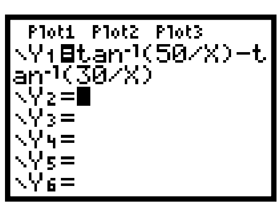
$\theta = \tan^{-1}\dfrac{50}{x} - \tan^{-1}\dfrac{30}{x}$

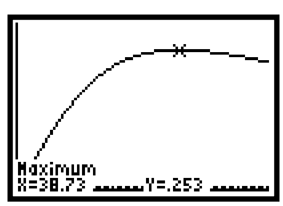
b) Write an equation for $\dfrac{d\theta}{dx}$

$\dfrac{d\theta}{dx} = \dfrac{-50}{x^2+2500} + \dfrac{30}{x^2+900}$



c) The sign is easiest to read at the value of x where θ stops increasing and starts decreasing. This happens when $\dfrac{d\theta}{dx} = 0$. Find x and confirm using the calculator. $5x^2 + 4500 = 3x^2 + 7500 \Rightarrow x = 38.730$ ft.





Inverse Trig Functions Integration - Classwork

If u is a differentiable function of x, and $a > 0$ then

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + C$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

1. $\int \frac{dx}{\sqrt{1 - x^2}}$

$\boxed{\sin^{-1} x + C}$

2. $\int \frac{dx}{\sqrt{4 - x^2}}$

$a = 2, u = x, du = dx$

$\sin^{-1}\frac{x}{2} + C$

3. $\int \frac{dx}{1 + x^2}$

$\boxed{\tan^{-1} x + C}$

4. $\int \frac{dx}{\sqrt{5 - 2x^2}}$

$a = \sqrt{5}, u = x\sqrt{2}, du = \sqrt{2}dx$

$\frac{1}{\sqrt{2}}\sin^{-1}\frac{x\sqrt{2}}{\sqrt{5}} = \frac{1}{\sqrt{2}}\sin^{-1}\frac{x\sqrt{10}}{5} + C$

5. $\int \frac{x\ dx}{\sqrt{5 - 2x^2}}$

$u = 5 - 2x^2,\ du = -4x\ dx$

$\frac{-1}{4}\int \frac{-4x\ dx}{\sqrt{5 - 2x^2}} = \frac{-1}{4}\int u^{-1/2}du$

$\frac{-1}{4}\left(2u^{1/2}\right) = \frac{-\sqrt{5 - 2x^2}}{2} + C$

6. $\int \frac{e^x}{9 + e^x}dx$

$u = e^x \qquad du = e^x dx$

$\ln\left|9 + e^x\right| + C$

7. $\int \frac{x + 2}{\sqrt{4 - x^2}}dx$

$\int \frac{x}{\sqrt{4 - x^2}}dx + \int \frac{2}{\sqrt{4 - x^2}}dx$

$-\sqrt{4 - x^2} + 2\sin^{-1}\frac{x}{2} + C$

8. $\int \frac{x + 2}{x^2 + 4}dx$

$\int \frac{x}{x^2 + 4}dx + \int \frac{2}{x^2 + 4}dx$

$\frac{\ln\left(x^2 + 4\right)}{2} + \tan^{-1}\frac{x}{2} + C$

9. $\int \frac{x^3}{x^2 + 1}dx$

$\int \left(x - \frac{x}{x^2 + 1}\right)dx$

$\frac{x^2}{2} - \frac{\ln\left(x^2 + 1\right)}{2} + C$

Inverse Trig Functions Integration - Homework

1. $\int \frac{dx}{1+4x^2}$

$a=1, u=2x, du=2dx$

$\frac{1}{2}\tan^{-1}2x+C$

2. $\int \frac{dx}{\sqrt{4-x^2}}$

$a=2, u=x, du=dx$

$\sin^{-1}\frac{x}{2}+C$

3. $\int \frac{dx}{4+(x-1)^2}$

$a=2, u=x-1, du=dx$

$\frac{1}{2}\tan^{-1}\frac{(x-1)}{2}+C$

4. $\int \frac{t}{\sqrt{1-t^4}}\,dt$

$a=1, u=t^2, du=2tdt$

$\frac{1}{2}\sin^{-1}t^2+C$

5. $\int \frac{x}{x^4+16}\,dx$

$a=4, u=x^2, du=2xdx$

$\frac{1}{8}\tan^{-1}\frac{x^2}{4}+C$

6. $\int \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

$u=\sin^{-1}x \quad du=\frac{1}{\sqrt{1-x^2}}\,dx$

$\frac{(\sin^{-1}x)^2}{2}+C$

7. $\int \frac{\sin x}{1+\cos^2 x}\,dx$

$a=1, u=\cos x, du=-\sin x dx$

$-\tan^{-1}(\cos x)+C$

8. $\int \frac{e^{2x}}{9+e^{4x}}\,dx$

$a=3, u=e^{2x}, du=2e^{2x}dx$

$\frac{1}{6}\tan^{-1}\left(\frac{e^{2x}}{3}\right)+C$

9. $\int_0^{\sqrt{3}/2} \frac{1}{1+4x^2}\,dx$

$a=1, u=2x, du=2dx$

$\frac{1}{2}\tan^{-1}(2x)\Big|_0^{\sqrt{3}/2}=\tan^{-1}\sqrt{3}=\frac{\pi}{6}$

10. Find the area of the region bounded by the curves $y=\frac{1}{\sqrt{4-x^2}},\ y=0,\ x=0,\ x=1$

$\sin^{-1}\frac{x}{2}\Big|_0^1=\frac{\pi}{6}$

Derivatives of Inverse Functions - Classwork

General Problem: Find the derivative of the inverse function of $f(x)$ at $x = k$.

Method 1: Simply finding the inverse function. This works when it is easy to generate the inverse function.

a) Find the inverse function by interchanging x and y and solving for y
b) Take the derivative of this new y. That will be the derivative of the inverse function.
c) Plug in your given k value

Method 2: Not finding the inverse function because it is too difficult

a) Find the inverse function by interchanging x and y
b) find $\frac{dy}{dx}$ implicitly
c) Solve for $\frac{dy}{dx}$. It will be in terms of y.
d) Replace the value of k for x in your inverse function from step a) above and solve for y
e) Plug that value of y into $\frac{dy}{dx}$

Example: If $f(x) = x^2, x \ge 0$, find the derivative of $f^{-1}(x)$ at $x = 4$.

METHOD 1

a) $y = x^2$, so the inverse is $x = y^2$

therefore $y = \sqrt{x}$ (first quadrant)

b) $y' = \frac{1}{2\sqrt{x}}$

c) $y'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

METHOD 2

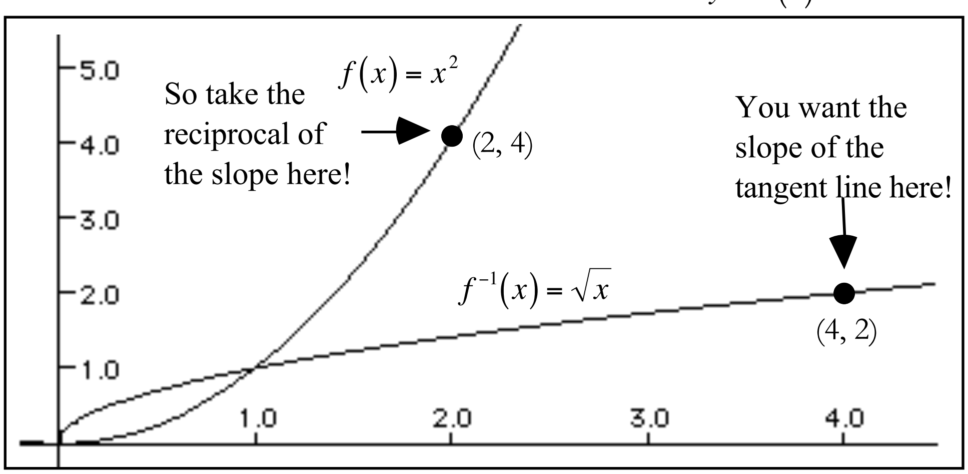
a) $y = x^2$, so the inverse is $x = y^2$

b) $1 = 2y\frac{dy}{dx}$

c) $\frac{dy}{dx} = \frac{1}{2y}$

d) $4 = y^2 \Rightarrow y = 2$ (quad I)

e) $\frac{dy}{dx} = \frac{1}{2y} = \frac{1}{2(2)} = \frac{1}{4}$



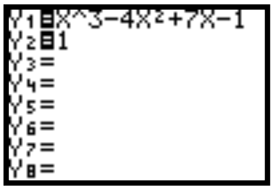
Note: It was necessary to restrict the domain of $f(x)$ to $x \ge 0$ so that its inverse is a function: i.e. that $f(x)$ is one-to-one (passes the horizontal line test).

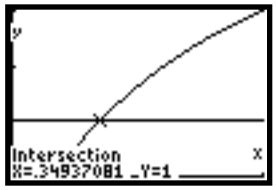
Example: Find the derivative of the inverse function of $f(x) = x^3 - 4x^2 + 7x - 1$ at $x = 1$.

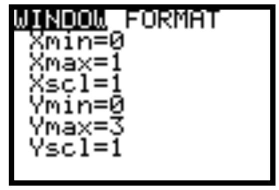
Method 1 will be too difficult. $y = x^3 - 4x^2 + 7x - 1$ so the inverse is $x = y^3 - 4y^2 + 7y - 1$

a) $1 = (3y^2 - 8y + 7)\dfrac{dy}{dx} \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{3y^2 - 8y + 7}$

b) Set $y^3 - 4y^2 + 7y - 1 = 1$. Graphically, you get $y = 0.349$.







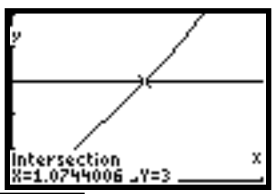
c) $\left.\dfrac{dy}{dx}\right|_{(y=.349)} = \dfrac{1}{3(.349)^2 - 8(.349) + 7} = .219$

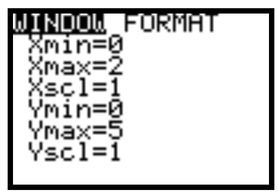
Example: Find the derivative of the inverse function of $f(x) = e^x + \ln x$ at $x = 3$.

Method 1 will be too difficult. $y = e^x + \ln x$ so the inverse is $x = e^y + \ln y$

a) $1 = \left(e^y + \dfrac{1}{y}\right)\dfrac{dy}{dx} \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{e^y + \dfrac{1}{y}}$ or $\dfrac{dy}{dx} = \dfrac{y}{ye^y + 1}$

b) Set $e^y + \ln y = 3$. Graphically, you get $y = 1.074$.





c) $\left.\dfrac{dy}{dx}\right|_{(y=1.074)} = \dfrac{1}{e^{1.074} + \dfrac{1}{1.074}} = .259$

Note: After you graphically intersect, you can easily get the answer by $\dfrac{1}{\text{nDeriv(Y1,X,X)}}$

Example: Find the derivative of the inverse function of $y = e^{x^2}, x > 0$.

Solution:

Inverse Function: $x = e^{y^2}$

$\ln x = y^2$

$y = \sqrt{\ln x} = (\ln x)^{1/2} \qquad \dfrac{dy}{dx} = \dfrac{1}{2}(\ln x)^{-1/2}\dfrac{1}{x} = \dfrac{1}{2x(\ln x)^{1/2}}$

Sample Problems: Find the derivative of the inverse function of (use method 2 only if method 1 won't work)

a) $y = x^3 + 1$ at $x = 9$ Method I

Inv: $y = \sqrt[3]{x - 1} \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{3(x-1)^{2/3}}$

$\left.\dfrac{dy}{dx}\right|_{(x=8)} = \dfrac{1}{12}$

b) $y = x^3 + 5x - 1$ at $x = 5$ Meth. II

Inv: $x = y^3 + 5y - 1 = 5 \Rightarrow y = 1$

$\left.\dfrac{dy}{dx}\right|_{(y=1)} = \dfrac{1}{3y^2 + 5} = \dfrac{1}{8}$

c) $y = x + \sin x$ at $x = \pi$ Meth. II

Inv: $x = y + \sin y = \pi \Rightarrow y = \pi$

$\left.\dfrac{dy}{dx}\right|_{(y=\pi)} = \dfrac{1}{1 + \cos y} = DNE$

Derivatives of Inverse Functions - Homework

For the problems below, find the derivative of f^{-1} for the function f at the specified value of x. No calculators.

1. $f(x)=x^3+2x-1$ at $x=2$

$$\text{Inv}: x=y^3+2y-1=2 \Rightarrow y=1$$
$$\left.\frac{dy}{dx}\right|_{(y=1)}=\frac{1}{3y^2+2}=\frac{1}{5}$$

2. $f(x)=2x^5+x^3+1$ at $x=4$

$$\text{Inv}: x=2y^5+y^3+1=4 \Rightarrow y=1$$
$$\left.\frac{dy}{dx}\right|_{(y=1)}=\frac{1}{10y^4+3y^2}=\frac{1}{13}$$

3. $f(x)=\sin x \quad -\frac{\pi}{2}\le x\le\frac{\pi}{2}$ at $x=\frac{1}{2}$

$$\text{Inv}: x=\sin y \Rightarrow y=\sin^{-1}x$$
$$\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}} \Rightarrow \left.\frac{dy}{dx}\right|_{(x=\pi/6)}=\frac{2}{\sqrt{3}} \quad \text{or}$$
$$\text{Inv}: x=\sin y=\frac{1}{2} \Rightarrow y=\frac{\pi}{6}$$
$$\left.\frac{dy}{dx}\right|_{\left(y=\frac{\pi}{6}\right)}=\frac{1}{\cos y}=\frac{2}{\sqrt{3}}$$

4. $f(x)=\cos 2x \quad -\frac{\pi}{2}\le x\le\frac{\pi}{2}$ at $x=1$

$$\text{Inv}: x=\cos 2y \Rightarrow y=\frac{\cos^{-1}x}{2}$$
$$\left.\frac{dy}{dx}\right|_{(x=1)}=\frac{-1}{2\sqrt{1-x^2}} \Rightarrow \left.\frac{dy}{dx}\right|_{(x=\pi/6)} \text{DNE} \quad \text{or}$$
$$\text{Inv}: x=\cos 2y=1 \Rightarrow y=0$$
$$\left.\frac{dy}{dx}\right|_{(y=0)}=\frac{-1}{2\sin 2y}=\text{DNE}$$

5. $f(x)=x^3-\frac{4}{x} \quad x>0$ at $x=6$

$$\text{Inv}: x=y^3-\frac{4}{y}=6 \Rightarrow y=2$$
$$\left.\frac{dy}{dx}\right|_{(y=2)}=\frac{1}{3y^2+\frac{4}{y^2}}=\frac{1}{13}$$

6. $f(x)=\sqrt{x-4}$ at $x=2$

$$\text{Inv}: x=\sqrt{y-4} \Rightarrow y=x^2+4$$
$$\left.\frac{dy}{dx}\right|_{(x=2)}=2x \Rightarrow \left.\frac{dy}{dx}\right|_{(x=2)} 4 \quad \text{or}$$
$$\text{Inv}: x=\sqrt{y-4}=2 \Rightarrow y=8$$
$$\left.\frac{dy}{dx}\right|_{(y=8)}=\frac{1}{\frac{1}{2\sqrt{y-4}}}=4$$

For the problems below, find the derivative of f^{-1} for the function f at the specified value of x. Use calculators.

7. $f(x)=x^3-2x^2+5x-1$ at $x=2$

$$\text{Inv}: x=y^3-2y^2+5y-1=2 \Rightarrow y=.737$$
$$\left.\frac{dy}{dx}\right|_{(y=.737)}=\frac{1}{3y^2-4y+5}=.272$$

8. $f(x)=\sqrt[3]{3x-5}$ at $x=-3$

$$\text{Inv}: x=\sqrt[3]{3y-5}=-3 \Rightarrow y=-7.333$$
8. $$\left.\frac{dy}{dx}\right|_{(y=-7.333)}=\frac{1}{\frac{1}{(3y-5)^{2/3}}}=9$$

9. $f(x)=\frac{x}{2}+\sin^2 x$ at $x=3$

$$\text{Inv}: x=\frac{y}{2}+\sin^2 y=3 \Rightarrow y=4.309$$
$$\left.\frac{dy}{dx}\right|_{(y=4.309)}=\frac{1}{\frac{1}{2}+2\sin y\cos y}=0.818$$

10. $f(x)=xe^{\cos x}$ at $x=3$

$$\text{Inv}: x=ye^{\cos y}=3 \Rightarrow y=4.335$$
10. $$\left.\frac{dy}{dx}\right|_{(y=4.335)}=\frac{1}{ye^{\cos y}(-\sin y)+e^{\cos y}}=0.287$$

Differential Equations by Separation of Variables - Classwork

A differential equation will be in the form of $\frac{dy}{dx} = f(x) \cdot g(y)$. In order to solve it, you must put it in the form of $g(y) \cdot dy = f(x) \cdot dx$ allowing you to integrate. Your goal is to get an equation in the form of $y = h(x)$

1) $\frac{dy}{dx} = \frac{2x}{y}$
$\int y\, dy = \int 2x\, dx$
$\frac{y^2}{2} = x^2 + C$
$y = \pm\sqrt{2x^2 + C}$

2) $\frac{dy}{dx} = y^2$
$\int y^{-2} dy = \int dx$
$\frac{y^{-1}}{-1} = x + C$
$y = \frac{-1}{x + C}$

3) $\frac{dy}{dx} = \frac{x + \sin(x)}{3y^2}$
$\int 3y^2 dy = \int [x + \sin(x)]\, dx$
$y^3 = \frac{x^2}{2} - \cos x + C$
$y = \sqrt[3]{\frac{x^2}{2} - \cos x + C}$

4) $\frac{dy}{dx} = 4y$
$\int \frac{dy}{y} = \int 4\, dx$
$\ln|y| = 4x + C$
$y = e^{4x+C}$
$y = Ce^{4x}$

5. $\frac{dy}{dx} = ky$
$\int \frac{dy}{y} = \int k\, dx$
$\ln|y| = kx + C$
$y = e^{kx+C}$
$y = Ce^{kx}$

6) $\frac{dy}{dx} = xy$
$\int \frac{dy}{y} = \int x\, dx$
$\ln|y| = \frac{x^2}{2} + C$
$y = e^{\frac{x^2}{2}+C} = Ce^{\frac{x^2}{2}}$

7. $\frac{du}{dt} = e^{u+2t} = e^u \cdot e^{2t}$
$\int \frac{du}{e^u} = \int e^{2t}\, dt$
$-e^{-u} = \frac{1}{2}e^{2t} + C \Rightarrow u = -\ln\left|-\frac{1}{2}e^{2t} + C\right|$

8. $\frac{dx}{dt} = 1 + t - x - tx = (1+t)(1-x)$
$\int \frac{dx}{1-x} = \int (1+t)\, dt$
$-\ln|1-x| = t + \frac{t^2}{2} + C$
$1 - x = e^{-\frac{t^2}{2}-t+C} \Rightarrow x = 1 + Ce^{-\frac{t^2}{2}-t}$

Find the solution of the differential equation that satisfies the given condition.

9. $xe^{-t}\frac{dx}{dt} = 1,\ x(0) = 1$
$\int x\, dx = \int e^t dt \Rightarrow \frac{x^2}{2} = e^t + C$
$\frac{1}{2} = 1 + C \Rightarrow C = \frac{-1}{2} \Rightarrow x^2 = 2e^t - 1$

10. $\frac{dy}{dx} = \frac{1+x}{xy},\ y(1) = -4$
$\int y\, dy = \int \frac{1+x}{x}\, dx \Rightarrow \frac{y^2}{2} = \ln x + x + C$
$8 = 1 + C \Rightarrow C = 7 \Rightarrow y^2 = 2\ln x + 2x + 14$

11. $\frac{dy}{dx} = y^2 + 1,\ y(1) = 0$
$\int \frac{1}{y^2+1}\, dy = \int dx \Rightarrow \tan^{-1} y = x + C$
$y = \tan(x + C) \Rightarrow c + -1 \Rightarrow y = \tan(x - 1)$

12. $x + 2y\sqrt{x^2+1}\frac{dy}{dx} = 0,\ y(0) = 1$
$2\int y\, dy = \int \frac{-x}{\sqrt{x^2+1}}\, dx \Rightarrow y^2 = C - \sqrt{x^2+1}$
$1 = C - 1 \Rightarrow C = 1 \Rightarrow y^2 = 2 - \sqrt{x^2+1}$

Differential Equations by Separation of Variables - Homework

1. $\frac{dy}{dx} = \frac{x}{y}$

$\int y\,dy = \int x\,dx \Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + C$

$y^2 = x^2 + C$

2. $\frac{dy}{dx} = \frac{x^2+2}{3y^2}$

$\int 3y^2\,dy = \int \left(x^2+2\right)dx$

$y^3 = \frac{x^3}{3} + 2x + C$

3. $x\frac{dy}{dx} = y$

$\int \frac{dy}{y} = \int \frac{dx}{x}$

$\ln|y| = \ln|x| + C$

$y = Cx$

4. $(2+x)\frac{dy}{dx} = 3y$

$\int \frac{dy}{y} = \int \frac{dx}{2+x}$

$\ln|y| = 3\ln|2+x| + C$

$y = C(2+x)^3$

5. $yy' = \sin x$

$\int y\,dy = \int \sin x dx$

$\frac{y^2}{2} = -\cos x + C$

$y^2 = -2\cos x + C$

6. $\left(1+4x^2\right)y' = 1$

$\int dy = \int \frac{1}{\left(1+4x^2\right)}dx$

$y = \frac{1}{2}\tan^{-1}2x + C$

Find the solution of the differential equation that satisfies the given condition.

7. $\sqrt{x} + \sqrt{y}\frac{dy}{dx} = 0\ ,\ y(1) = 4$

$\int \sqrt{y}\,dy = \int -\sqrt{x}dx \Rightarrow \frac{2}{3}y^{3/2} = -\frac{2}{3}x^{3/2} + C$

$y^{3/2} = -x^{3/2} + C \Rightarrow 8 = -1 + C \Rightarrow C = 9$

$y^{3/2} = -x^{3/2} + 9$

8. $y\frac{dy}{dx} = e^x\ ,\ y(0) = 4$

$\int y\,dy = \int e^x dx \Rightarrow \frac{y^2}{2} = e^x + C$

$8 = 1 + C \Rightarrow c = 7 \Rightarrow y^2 = 2e^x + 14$

9. $xy\frac{dy}{dx} - \ln x = 0\ ,\ y(1) = 0$

$\int y\,dy = \int \frac{\ln x}{x}dx \Rightarrow \frac{y^2}{2} = \frac{(\ln x)^2}{2} + C$

$0 = 0 + C \Rightarrow C = 0 \Rightarrow y = \pm(\ln x)$

10. $y(x+1) + \frac{dy}{dx} = 0\ ,\ y(-2) = 1$

$\int \frac{dy}{y} = \int (-x-1)dx \Rightarrow \ln|y| = \frac{-x^2}{2} - x + C$

$y = Ce^{\frac{-x^2}{2}-x} \Rightarrow 1 = Ce^0 \Rightarrow y = e^{\frac{-x^2}{2}-x}$

11. $\left(1+x^2\right)\frac{dy}{dx} - \left(1+y^2\right) = 0\ ,\ y(0) = \sqrt{3}$

$\int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2} \Rightarrow \tan^{-1}y = \tan^{-1}x + C$

$\frac{\pi}{3} = C \Rightarrow y = \tan\left(\tan^{-1}x + \frac{\pi}{3}\right)$

12. $dT + k(T-70)dt = 0\ ,\ T(0) = 140$

$\int \frac{dT}{T-70} = \int -k\,dt \Rightarrow \ln|T-70| = -kt + C$

$T - 70 = Ce^{-kt} \Rightarrow 70 = C$

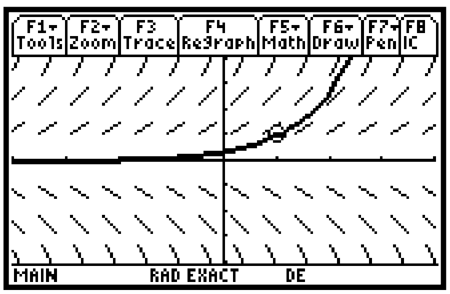
$T = 70 + 70e^{-kt}$

Slope Fields - Classwork

In this section, we can solve differential equations by obtaining a *slope field* (sometimes called a *direction field*) that approximates the general solution. The slope field of a first-order differential equation says that the differential equation can be interpreted as a statement about the slopes of its solution curves. You are given dy/dx.

1) $\dfrac{dy}{dx} = y$. Fill in the chart for $\dfrac{dy}{dx}$

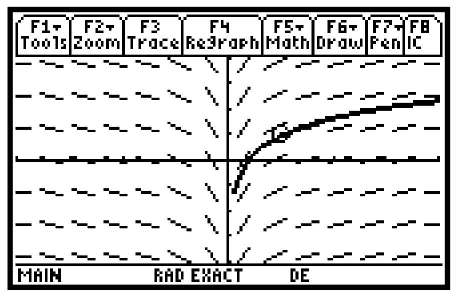
(x, y)	-3	-2	-1	0	1	2	3
3	3	3	3	3	3	3	3
2	2	2	2	2	2	2	2
1	1	1	1	1	1	1	1
0	0	0	0	0	0	0	0
-1	-1	-1	-1	-1	-1	-1	-1
-2	-2	-2	-2	-2	-2	-2	-2
-3	-3	-3	-3	-3	-3	-3	-3



Find the solution going through
a) (1, 1) b) (-2, 0)

2) $\dfrac{dy}{dx} = \dfrac{1}{x}$ Fill in the chart for $\dfrac{dy}{dx}$

(x, y)	-3	-2	-1	0	1	2	3
3	.-33	-.5	-1	∞	1	.5	.333
2	.-33	-.5	-1	∞	1	.5	333
1	.-33	-.5	-1	∞	1	.5	333
0	.-33	-.5	-1	∞	1	.5	333
-1	.-33	-.5	-1	∞	1	.5	333
-2	.-33	-.5	-1	∞	1	.5	333
-3	.-33	-.5	-1	∞	1	.5	333

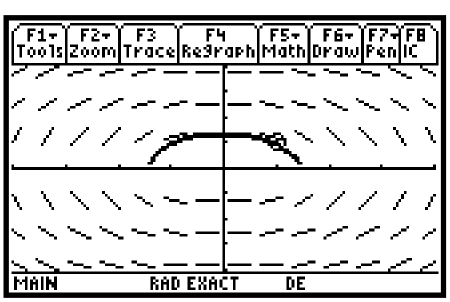


Find the solution going through
a) (1, 1) b) (-2, -1)

---

3) $\dfrac{dy}{dx} = \dfrac{-x}{y}$ Fill in the chart for $\dfrac{dy}{dx}$

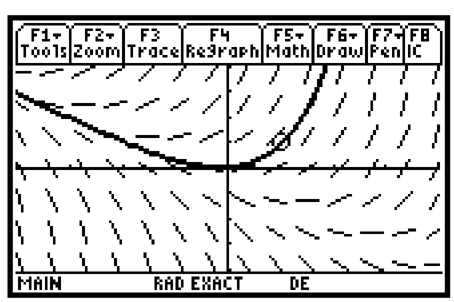
(x, y)	-3	-2	-1	0	1	2	3
3	1	.67	.33	0	-.33	-.67	-1
2	1.5	1	.5	0	-.5	-1	-1.5
1	.3	2	1	0	-1	-2	-3
0	∞	∞	∞	∞	∞	∞	∞
-1	-3	-2	-1	0	1	2	3
-2	-.1.5	-1	-.5	0	.5	1	1.5
-3	-1	-.67	-.33	0	.33	.67	1



Find the solution going through
a) (1, 1) b) (-2, 0)

4) $\dfrac{dy}{dx} = x + y$ Fill in the chart for $\dfrac{dy}{dx}$

(x, y)	-3	-2	-1	0	1	2	3
3	0	1	2	3	4	5	6
2	-1	0	1	2	3	4	5
1	-2	-1	0	1	2	3	4
0	-3	-2	-1	0	1	2	3
-1	-4	-3	-2	-1	0	1	2
-2	-5	-4	-3	-2	-1	0	1
-3	-6	-5	-4	-3	-2	-1	0

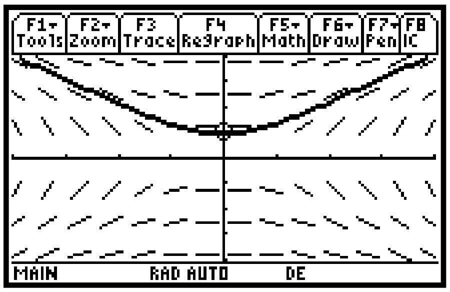


Find the solution going through
a) (1, 1) b) (-2, -1)

Slope Fields - Homework

1. $\frac{dy}{dx} = \frac{x}{y}$ Fill in the chart for $\frac{dy}{dx}$.

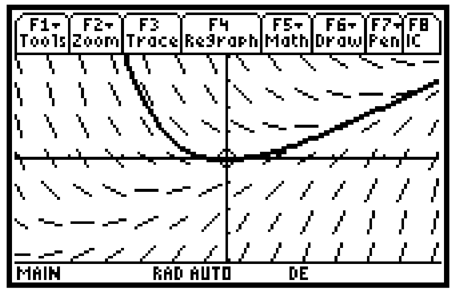
(x, y)	-3	-2	-1	0	1	2	3
3	-1	-.67	-.33	0	.33	.67	1
2	-1.5	-1	-.5	0	.5	1	1.5
1	-3	-2	-1	0	1	2	3
0	∞	∞	∞	∞	∞	∞	∞
-1	.3	2	1	0	-1	-2	-.3
-2	1.5	1	.5	0	.-5	-1	-1.5
-3	1	.67	.33	0	-.33	-.67	-1



Find the solution going through
a) (0, 1) b) (1, 0) c) (0, 3)

2. $\frac{dy}{dx} = x - y$ Fill in the chart for $\frac{dy}{dx}$

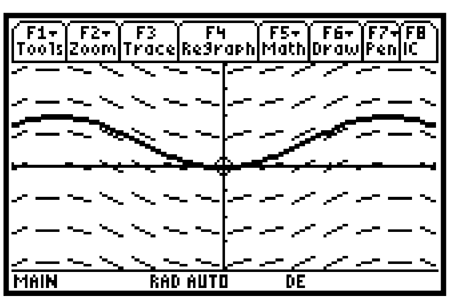
(x, y)	-3	-2	-1	0	1	2	3
3	-6	-5	-4	-3	-2	-1	0
2	-5	-4	-3	-2	-1	0	1
1	-4	-3	-2	-1	0	1	2
0	-3	-2	-1	0	1	2	3
-1	-2	-1	0	1	2	3	4
-2	-1	0	1	2	3	4	5
-3	0	1	2	3	4	5	6



Find the solution going through
a) (0, 0) b) (0, -1) c) (1, -2)

3. $\frac{dy}{dx} = \sin x$ Fill in the chart for $\frac{dy}{dx}$.

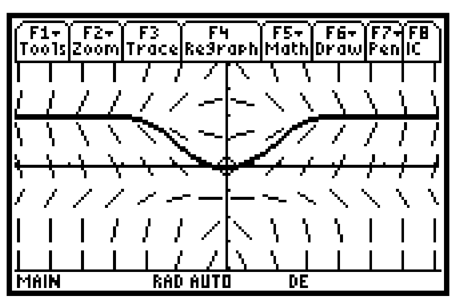
(x, y)	-3	-2	-1	0	1	2	3
3	-0.14	-0.91	-0.84	0.00	0.84	0.91	0.14
2	-0.14	-0.91	-0.84	0.00	0.84	0.91	0.14
1	-0.14	-0.91	-0.84	0.00	0.84	0.91	0.14
0	-0.14	-0.91	-0.84	0.00	0.84	0.91	0.14
-1	-0.14	-0.91	-0.84	0.00	0.84	0.91	0.14
-2	-0.14	-0.91	-0.84	0.00	0.84	0.91	0.14
-3	-0.14	-0.91	-0.84	0.00	0.84	0.91	0.14



Find the solution going through
a) (0, 0) b) (1, 0)

4. $\frac{dy}{dx} = x(1+y)(2-y)$ Fill in the chart for $\frac{dy}{dx}$.

(x, y)	-3	-2	-1	0	1	2	3
3	12	8	4	0	-4	-8	-12
2	0	0	0	0	0	0	0
1	-6	-4	-2	0	2	4	6
0	-6	4	-2	0	2	4	6
-1	0	0	0	0	0	0	0
-2	12	8	4	0	-4	-8	-12
-3	30	20	10	0	-10	-20	-30



Find the solution going through
a) (0, 0) b) (0, 1)

Exponential Growth - Classwork

Consider the statement "The rate of change of some quantity y is directly proportional to y"

This is like saying that the more money you have (y), the faster it will grow $\left(\frac{dy}{dt}\right)$, or the more you scratch an insect bite, the worse it will get, or the more addicted someone is to a substance, the more the addiction will grow.

"The rate of change of some quantity y is directly proportional to y" can be translated thusly:

quantity = y **rate of change of y** = $\frac{dy}{dt}$ **directly proportional** = multiplied by some constant k

So this statement can be translated: $\frac{dy}{dt} = ky$ k is sometimes called the growth (or decay) constant

This is a differential equation and can be solved:

$\frac{dy}{y} = k\,dt$	Separate the variables
$\int \frac{dy}{y} = \int k\,dt$	Integrate both sides
$\ln\lvert y\rvert = k\,t + C$	Perform the integration. Put the $+C$ on the right side
$\lvert y\rvert = e^{k\,t+C}$	Write this exponentially
$y = e^{k\,t+C}$	Drop the absolute value as e raised to any power > 0
$y = e^{k\,t} \cdot e^{C}$	Split the right side into 2 factors
$y = Ce^{k\,t}$	e^{C} is simply a constant
$y = y_0 e^{k\,t}$	When $t = 0$, $y = C$. We call the constant y_0

So, the statement: the rate of change of some quantity y is directly proportional to y is translated:

$\frac{dy}{dt} = ky$ which can in turn be translated into $y = Ce^{kt}$ or $y = y_0 e^{kt}$

KNOW THIS BACKWARD AND FORWARD

Example 1: *Punctured Tire Problem*: You run over a nail. As the air leaks out of your tire, the rate of change of air pressure inside the tire is directly proportional to that pressure.

a. Write a differential equation that states that fact if the pressure is 35 lbs/psi and decreasing at the rate of .28 lbs/psi/min at the time the nail is struck.

$$\frac{dP}{dt} = kP$$
$$-.28 = 35k \Rightarrow k = -.008$$

b. Solve that differerential equation.

$$\frac{dP}{dt} = -.008P$$
$$P = 35e^{-.008t}$$

c. What will the pressure be at 10 minutes after the tire was punctured.

$$P = 35e^{-.008(10)} = 32.309 \text{ lbs/psi}$$

d. The car is safe to drive as long as the pressure is 12 lbs/psi or greater. For how long after the puncture will the car be safe to drive?

$$12 = 35e^{-.008t} \Rightarrow \frac{12}{35} = e^{-.008t} \Rightarrow \ln\left(\frac{12}{35}\right) = -.008t$$
$$t = \frac{\ln\left(\frac{12}{35}\right)}{-.008} = 133.805 \text{ min}$$

Example 2: Bacteria in a lab culture grows in such a way that the instantaneous rate of change of bacteria is directly proportional to the number of bacteria present.

a) Write a differential equation that expresses the relationship.

$$\frac{dP}{dt} = kP$$

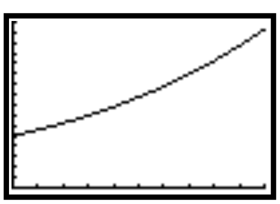
b) Solve the differential equation for the number of bacteria as a function of time.

$$P = Ce^{kt}$$

c) Suppose that initially there are 5 million bacteria and three hours later, that number has grown to 7 million. Write the particular equation that expresses the number of millions of bacteria as a function of hours.

$$P = Ce^{kt} \Rightarrow P = 5e^{kt}$$
$$7 = 5e^{3k} \Rightarrow k = \frac{\ln 1.4}{3} = .112 \Rightarrow P = 5e^{.112t}$$

d) Sketch the graph of time vs. bacteria over 10 hours



e) What will the bacteria population be one full day after the first measurement?

$$5e^{.112(24)} = 73.789 \text{ million}$$

f) When will the population reach 1 billion? Show your work

$$5e^{.kt} = 1000$$
$$t = \frac{\ln 200}{k} = 47.240 \text{ hours}$$

Exponential Growth - Homework

1. Major purchases like cars depreciate in value. That is, as time goes on, their value goes down. So the change in a car's price is directly proportional to its current value.

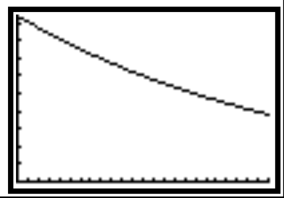
   a) Write a differential equation that expresses this relationship, and then solve the DEQ for the value of the car.

   $$\frac{dP}{dt} = kP \Rightarrow P = Ce^{kt}$$

   b) Suppose you own a car whose trade-in value is presently \$4200. Three months ago, its trade-in value was \$4700. Find the particular equation that expressed the trade-in value since the car was worth \$4700.

   $$P = 4700e^{kt} \Rightarrow 4200 = 4700e^{3k} \Rightarrow k = \ln\left(\frac{42}{47}\right)\Big/3 = -.037$$

   c) Sketch the graph of this equation over 2 years.



   d) What will be the trade-in value of the car one year after the car was worth \$4700?

   e) You plan to get rid of the car when its trade-in value drops to \$1200. When will this be?

   $P(12) = 2997.12$

   $1200 = 4700e^{kt}$

   $t = \ln\left(\frac{12}{47}\right)\Big/k \approx 36.414$ months

   happens in the 37th month

   f) At the time your car was worth \$4700, it was 31 months old. What was its trade-in value when it was new?

   g) The purchase price of the car when it was new was \$16,000. How do you explain the difference between this number and f)?

   $P(-31) = 15{,}026.80$ Car loses value when it is driven.

2. Carbon 14 is an isotope of carbon that is formed when radiation from the sun strikes ordinary carbon dioxide in the atmosphere. Thus plants such as trees, which get their carbon dioxide from the atmosphere, contains small parts of carbon 14. The rate of carbon 14 decay is directly proportional to the amount present.

   a) Write a differential equation that expresses this relationship where P is the percentage of carbon 14 that remains in a tree that grew t years ago.

   $$\frac{dP}{dt} = kP \Rightarrow P = Ce^{kt}$$

   b) Solve the differential equation for P in terms of t. Use the fact that the half-life of carbon 14 is 5750 years to solve the particular equation for the constant P_0. (If P =100 when t = 0, then P = 50 when t = 5750).

   $$50 = 100e^{5750k} \Rightarrow k = \ln .5/5750$$

c) The oldest living trees in the world are the bristlecone pines in the White Mountains of California. 4000 growth rings have been counted in the trunk of one of these trees, meaning that the innermost ring grew 4000 years ago. What percentage of the original carbon 14 would you expect to find in this innermost ring?

$$\boxed{P(4000) = 61.74\%}$$

d) A piece of wood claimed to have come from Noah's Ark is found to have 48.37% of the carbon 14 remaining. It has been suggested that the Great Flood occurred in 4004 BC. Is the wood old enough to have come from Noah's Ark? Show why or why not.

$$\boxed{48.37 = 100e^{kt} \Rightarrow t = 6024 \text{ years ago} \Rightarrow \text{it could be old enough}}$$

3. The rate of change of atmospheric pressure *P* with respect to altitude h is proportional to *P*, provided that the temperature is constant. At 15°C, the pressure is 101.3 at sea level and 87.1 at height *h* = 1000m.

a) Write a differential equation that expresses this relationship where *P* is the pressure and *h* is the altitude. Then solve the particular DEQ given the information above.

$$\boxed{\frac{dP}{dh} = kP \Rightarrow P = 101.3e^{kh} \Rightarrow k = \ln\left(\frac{87.1}{101.3}\right)\Big/1000}$$

b) What is the pressure at the top of Mt. McKinley at an altitude of 6187 meters?

$$\boxed{P(6187) = 39.792}$$

c) At what altitude is the pressure equal to 50?

$$\boxed{50 = 101.3e^{kh} \Rightarrow h = \ln\left(\frac{50}{101.3}\right)\Big/k = 4675 \text{ m}}$$

4. You accidently inhale some poisonous fumes. Twenty hours later, you still feel woozy so you go to a doctor. From blood samples, he measures a poison concentration of .00372 mg/ml and tells you to come back in 8 hours. On the second visit, he measures a concentration of .00219 mg/ml.

Let *t* be the number of hours since your first doctor visit and *C* be the concentration of poison in your blood. The rate of change of *C* is directly proportional to the current value of *C*.

a) Write a DEQ that relates these two variables. $\boxed{\frac{dC}{dt} = kC}$

b) Solve the differential equation subject to initial conditions specified. Express *C* as a function of *t*.

$$\boxed{\frac{dC}{dt} = kC \Rightarrow P = .00372e^{kt} \Rightarrow k = \ln\left(\frac{.00219}{.00372}\right)\Big/8 = -.066}$$

c) The doctor says that you might have serious body damage if the poison concentration has ever been as high as 0.015 mg/ml. Based on your equation, was the concentration ever that high? Justify your answer.

Since $\frac{dC}{dt} < 0$, the maximum value of C must have been when the poison was taken, 20 hours before the visit to the doctor. $C(-20) = .0139$ - Never as high as 0.015.

5. The rate in which whooping cough spreads is proportional to the amount of whooping cough cases there presently is. In the course of any year, the number of whooping cough cases is reduced by 20%.

a) Write a DEQ that states the situation above and then solve the DEQ.

$$\boxed{80 = 100e^{k} \Rightarrow k = \ln .8 = -.223}$$

b) If there are 10,000 cases today, how many years will it take for the number of cases to reduce to 100?

$$\boxed{100 = 10000e^{kt} \Rightarrow t = \ln .01/k = 20.638 \text{ yrs}}$$

c) If the cases can be reduced by 25% instead of 20%, how long will it take to reduce to 100 cases?

$$\boxed{k = \ln .75 \Rightarrow 100 = 10000e^{kt} \Rightarrow t = \ln .01/k = 16.008 \text{ yrs}}$$

d) Using part c), how long will it take to eradicate the disease (reduce # of cases to less than 1)?

$$\boxed{1 = 10000e^{kt} \Rightarrow t = \ln .0001/k \approx 32 \text{ yrs}}$$

6. John Napier, who invented natural logarithms, was the first person to answer the question: If money compounds at a rate proportional to the amount you now have, and you invest money at 5% interest,

a) Write and solve the DEQ which states the above.

b) how long will it take \$100 to grow to \$1,000?

$$\frac{dM}{dt} = kM \Rightarrow M = M_0 e^{kt}$$

$$1.05 = 1.00e^{k} \Rightarrow k = \ln(1.05)$$

$$1000 = 100e^{kt}$$

$$t = \frac{\ln 10}{k} = 47.19 \text{ yrs}$$

c) how long will it take any amount of money to triple?

c) if you invest \$1,000, when will you be a millionaire?

$$3M = Me^{kt} \Rightarrow t = \ln 3/k = 22.517 \text{ yrs}$$

$$1000000 = 1000e^{kt}$$

$$t = \frac{\ln 1000}{k} = 141.581 \text{ yrs}$$

7. The processing of raw sugar has a step called "inversion" that changes the sugar's molecular structure. Once the process has begun, the rate of change of the amount of sugar is proportional to the amount of raw sugar remaining. If 1,000 kg of raw sugar reduces to 800 kg of raw sugar during the first 10 hours, how much sugar will remain after another 12 hours?

$$800 = 1000e^{10k} \Rightarrow k = \ln .8/10 = -.022$$

$$P = 1000e^{kt} \Rightarrow P(22) = 612.066 \text{ kg}$$

<u>Exponential Growth Continuation - Classwork</u>

Remember that you can always translate $\dfrac{dy}{dt} = ky$ **into** $y = Ce^{kt}$ if and only if the original statement is:
The rate of change of some quantity y is directly proportional to y. Suppose it isn't? For example:

a) the rate of change of y is proportional to $4y$.

$$\frac{dy}{dt} = 4ky \Rightarrow \int \frac{dy}{y} = \int 4k\,dt$$
$$\ln|y| = 4kt + C \Rightarrow y = Ce^{4kt}$$

b) the rate of change of y is proportional to $4 - y$.

$$\frac{dy}{dt} = k(4-y) \Rightarrow \int \frac{dy}{4-y} = \int k\,dt$$
$$-\ln|4-y| = kt + C \Rightarrow 4 - y = Ce^{kt}$$
$$y = 4 - Ce^{kt}$$

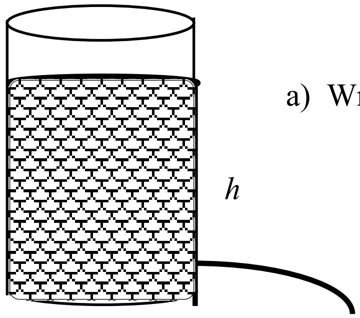
c) the rate of change of y is inversely proportional to y.

$$\frac{dy}{dt} = \frac{k}{y} \Rightarrow \int y\,dy = \int k\,dt$$
$$\frac{y^2}{2} = kt + C \Rightarrow y^2 = 2kt + C$$

d) the rate of change of y is proportional to $\sqrt{y}$

$$\frac{dy}{dt} = k\sqrt{y} \Rightarrow \int y^{-1/2}\,dy = \int k\,dt$$
$$2y^{1/2} = kt + C \Rightarrow y^{1/2} = \frac{kt}{2} + C$$
$$y = \left(\frac{kt}{2} + C\right)^2$$

Example. Suppose you fill a tall tin can with water and then punch out a hole near the bottom. The water leaks out quickly at first, then more slowly as the depth of the water decreases. In physics, it can be proved that the rate at which the water's height *h* changes (i.e. leaks out) is directly proportional to the square root of its height.



a) Write a differential equation which states this relationship.

$$\frac{dh}{dt} = k\sqrt{h}$$

b) Suppose that at time $t = 0$ min, the height is 12 cm and $dh/dt = -3$ cm/sec. Find the value of *k* which satisfies this relationship.

$$\frac{dh}{dt} = k\sqrt{h} \Rightarrow -3 = k\sqrt{12} \Rightarrow k = \frac{-\sqrt{3}}{2}$$

c) Solve this differential equation to find *h* as a function of *t*. Use the given information to find the particular solution. What kind of function is this?

$$\frac{dh}{dt} = -\frac{\sqrt{3}}{2}\sqrt{h} \Rightarrow \int h^{-1/2}\,dh = \int -\frac{\sqrt{3}}{2}\,dt \Rightarrow 2h^{1/2} = -\frac{\sqrt{3}}{2}t + C \Rightarrow h^{1/2} = -\frac{\sqrt{3}}{4}t + C$$
$$h = \left(-\frac{\sqrt{3}}{4}t + C\right)^2 \Rightarrow C = \sqrt{12} \Rightarrow h = \left(-\frac{\sqrt{3}}{4}t + \sqrt{12}\right)^2 \Rightarrow h = \frac{3}{16}t^2 - 3t + 12$$

d) Solve algebraically for the time when the can becomes empty.

$$-\frac{\sqrt{3}}{4}t + \sqrt{12} = 0 \Rightarrow \sqrt{3}t = 4\sqrt{12} \Rightarrow t = 8 \text{ sec}$$

Exponential Growth Continuation - Homework

1. A curve passes through the point (0, 5) and has the property that the slope of the curve at every point P is twice the y-coordinate of P. What is the equation of the curve?

$$\frac{dy}{dt} = 2y \Rightarrow \int \frac{dy}{y} = \int 2dt$$
$$\ln|y| = 2t + C \Rightarrow y = Ce^{2t} \Rightarrow y = 5e^{2t}$$

2. Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and the outside air. Suppose that a roast turkey is taken from the oven when its temperature has reached 185° and is placed on a table where the temperature is 75°. If R is the temperature of the turkey after t minutes, then Newton's Law of Cooling implies that:

$$\frac{dR}{dt} = k(R - 75)$$

a) Solve the differential equation for R. Then use the given information to find the particular solution.

$$\frac{dR}{dt} = k(R - 75) \Rightarrow \frac{dR}{(R - 75)} = k\,dt$$
$$\ln|R - 75| = kt + C \Rightarrow R - 75 = Ce^{kt}$$
$$R = 75 + Ce^{kt} \Rightarrow 185 = 75 + C \Rightarrow C = 110$$
$$R = 75 + 110e^{kt}$$

b) If the temperature of the turkey is 150° after half an hour, what is the temperature after 45 minutes?

$$150 = 75 + 110e^{30k} \Rightarrow k = \ln\frac{75}{110} \Big/ 30$$
$$R(45) = 136.929°$$

c) When will the turkey have cooled to 100°?

$$100 = 75 + 110e^{kt} \Rightarrow t = \ln\frac{25}{110} \Big/ k \Rightarrow k = 116.055 \text{ min}$$

3. The change of rate of coyotes N in a population is directly proportional to 650 - N. (where t is the time in years. When $t = 0$, the population is 300 and when $t = 2$, the population has increased to 500.

a) Write and solve a differential equation that describes this situation.

$$\frac{dN}{dt} = k(650 - N) \Rightarrow \frac{dN}{(650 - N)} = k\,dt \Rightarrow -\ln|650 - N| = kt + C \Rightarrow 650 - N = Ce^{-kt}$$
$$N = 650 - Ce^{-kt} \Rightarrow 300 = 650 - C \Rightarrow C = 350 \Rightarrow N = 650 - 350e^{-kt} \Rightarrow 500 = 650 - 350e^{-2k}$$
$$k = \ln\frac{150}{350} \Big/ -2 \approx .424$$

b. Find the coyote population in 3 years

c. Find $\lim_{t \to \infty} N(t)$

$N(3) \approx 552$ 650

4. Let $P(t)$ represent the number of students in a school who buy their lunch after *t* weeks. Suppose P is increasing at a rate proportionally to $600-P$ where the constant of proportionality is *k*.

a) Write the DEQ which states that fact. $\boxed{\dfrac{dP}{dt} = k(600-P)}$

b) If 300 students buy their lunch initially and 400 buy their lunches after 10 weeks, solve the DEQ.

$$\frac{dN}{dt} = k(600-N) \Rightarrow \int \frac{dP}{600-N} = \int k\,dt \Rightarrow -\ln|600-P| = kt + C \Rightarrow 600 - P = Ce^{-kt}$$
$$P = 600 - Ce^{-kt} \Rightarrow 300 = 600 - C \Rightarrow C = 300 \Rightarrow P = 600 - 300e^{-kt} \Rightarrow 400 = 600 - 300e^{-10k}$$
$$k = \ln\frac{200}{300}\Big/-10 \approx .041$$

c) How many students will buy their lunch after 20 weeks?

$$\boxed{P(20) \approx 467}$$

d) If school went on endlessly using this pattern, what is the limit to the number of students buying lunch?

$$\boxed{\lim_{t\to\infty} P(t) = 600}$$

5. You win a well-known national sweepstakes. Your award is \$100 a day for the rest of your life! You put the money in a bank where it earns interest at a rate directly proportional to the amount *M* which is in the account. So, $\dfrac{dM}{dt} = 100 + kM$ where *k* is the growth constant.

a) Solve the DEQ in general given the fact that at *t* = 0 days, there is no money in the account.

$$\frac{dM}{dt} = 100 + kM \Rightarrow \frac{1}{k}\int \frac{k\,dM}{100+kM} = \int dt \Rightarrow \frac{1}{k}\ln|100+kM| = t + C \Rightarrow 100 + kM = Ce^{kt}$$
$$kM = Ce^{kt} - 100 \Rightarrow M = \frac{Ce^{kt}-100}{k} \Rightarrow 0 = \frac{C-100}{k} \Rightarrow C = 100 \Rightarrow M = \frac{100e^{kt}-100}{k}$$

b) Suppose you invest the money at 5% APR So $k = \dfrac{.05}{365}$. Solve the DEQ completely.

$$\boxed{M(t) = \frac{365\left(100e^{.0001369t} - 100\right)}{.05}}$$

c) How much will you have at the end of a year?

$$\boxed{M(365) = \$37{,}427.90}$$

d) Assuming you live for 75 more years, how much will you take to the grave with you if you never spend it?

$$\boxed{M(75 \cdot 365) = \$30{,}310{,}389.86}$$

e) How long will it take you to become a millionaire? How about a billionaire? Calculators allowed.

$$\frac{100e^{kt}-100}{k} = 1000000 \text{ - Solve Graphically} \Rightarrow t = 6298.67 \text{ days} = 17.26 \text{ years}$$
$$\frac{100e^{kt}-100}{k} = 1000000000 \text{ - Solve Graphically} \Rightarrow t = 52729.33 \text{ days} = 144.46 \text{ years}$$

Finding "Impossible" Integrals - Classwork

Example: Let $F(x)$ be an antiderivative of $\dfrac{x}{x^3+1}$. If $F(0)=2$, find $F(3)$.

Solution: We need to take the integral of $\dfrac{x}{x^3+1}$. But can we? *u*-substitution does not work. We cannot split the function. There seems to be no possibility for inverse-trig. So how can we do the problem?

We are not being asked to find the indefinite integral of the expression. We just need to find the value of the integral function of this expression at 3.

So, if $F(x)=\int \dfrac{x}{x^3+1}\,dx$, then $F(3)-F(0)=\int_0^3 \dfrac{x}{x^3+1}\,dx$ and thus $F(3)=2+\int_0^3 \dfrac{x}{x^3+1}\,dx$.

Using the calculator: $F(3)=2+.879=2.879$.

2) Let $F(x)$ be an antiderivative of $\sqrt{4x^2+5}$. If $F(1)=3$, find $F(5)$.

$$F(5)-F(1)=\int_1^5 \sqrt{4x^2+5}\,dx$$

$$F(5)=F(1)+\int_1^5 \sqrt{4x^2+5}\,dx=3+25.866=28.866$$

3. Let $F(x)$ be an antiderivative of e^{x^2}. If $F(-1)=2$, find $F(-2)$.

$$F(-1)-F(-2)=\int_{-2}^{-1} e^{x^2}\,dx$$

$$F(-2)=F(-1)-\int_{-2}^{-1} e^{x^2}\,dx=2-14.990=-12.990$$

Finding "Impossible" Integrals - Homework

1. Let $F(x)$ be an antiderivative of $\dfrac{5x-2}{8x^4+2}$. If $F(1)=3$, find $F(6)$.

$$F(6)-F(1)=\int_1^6 \frac{5x-2}{8x^4+2}\,dx \Rightarrow F(6)=F(1)+\int_1^6 \frac{5x-2}{8x^4+2}\,dx=3+.206=3.206$$

2. Let $F(x)$ be an antiderivative of $\sqrt[3]{5x^2-4}$. If $F(2)=7$, find $F(5)$.

$$F(5)-F(2)=\int_2^5 \sqrt[3]{5x^2-4}\,dx \Rightarrow F(5)=F(2)+\int_2^5 \sqrt[3]{5x^2-4}\,dx=7+11.447=18.447$$

3. Let $F(x)$ be an antiderivative of $\sin^3 x$. If $F(-1)=4$, find $F(4)$

$$F(4)-F(-1)=\int_{-1}^4 \sin^3 x\,dx \Rightarrow F(4)=F(-1)+\int_{-1}^4 \sin^3 x\,dx=4+1.048=5.048$$

4. Let $F(x)$ be an antiderivative of $\ln(x^2+4x+12)$. If $F(10)=-2$, find $F(-1)$

$$F(10)-F(-1)=\int_{-1}^{10} \ln(x^2+4x+12)\,dx \Rightarrow F(-1)=F(10)-\int_{-1}^{10} \ln(x^2+4x+12)\,dx=-2-41.743=-43.743$$

L'Hopital's Rule for Indeterminate Forms - Classwork

There are 7 expressions which are considered to have an **indeterminate form.** That is -- we can not assign a definite value to the expression without further investigation. Perhaps there is a limit. Perhaps there is no limit (i.e. the limit is $\pm\infty$). These expressions are (when plugged in)

$$\frac{0}{0},\quad \frac{\infty}{\infty},\quad 0\cdot\infty,\quad 1^{\infty},\quad \infty-\infty.\quad \infty^{0}$$

We now have a way of evaluating any expression of the form $\dfrac{f(x)}{g(x)}=\dfrac{0}{0}=\dfrac{\infty}{\infty}$

L'Hopital's Rule states that if $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ then $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$

Type 1 - Simple L'Hopital's Rule for Polynomials and Powers

1. $\lim\limits_{x\to2}\left(\dfrac{3x^2-7x+2}{x-2}\right)=\lim\limits_{x\to2}\left(\dfrac{6x-7}{1}\right)=5$

2. $\lim\limits_{x\to-3}\dfrac{x+3}{\sqrt{x^2-5}-2}=\lim\limits_{x\to-3}\dfrac{\sqrt{x^2-5}}{x}=\dfrac{-2}{3}$

Note that L'Hopital's Rule will not work unless this expression is an indeterminate form. $\lim\limits_{x\to2}\left(\dfrac{2x+3}{x}\right)=\dfrac{7}{2}$, and not 2.

Type 2 - Repeated Use of L'Hopital's Rule for Polynomials

3. $\lim\limits_{x\to-2}\dfrac{x^3+x^2-8x-12}{x^3+8x^2+20x+16}=\lim\limits_{x\to-2}\dfrac{3x^2+2x-8}{3x^2+16x+20}$

$\lim\limits_{x\to-2}\dfrac{6x+2}{6x+16}=\dfrac{-10}{4}=-\dfrac{5}{2}$

4. $\lim\limits_{x\to1}\left(\dfrac{2x^4-7x^3+9x^2-5x+1}{2x^4-5x^3+3x^2+x-1}\right)=\dfrac{0}{0}$

$\lim\limits_{x\to1}\left(\dfrac{8x^3-21x^2+18x-5}{8x^3-15x^2+6x+1}\right)=\dfrac{0}{0}$

$\lim\limits_{x\to1}\left(\dfrac{24x^2-42x+18}{24x^2-30x+6}\right)=\dfrac{0}{0}$

$\lim\limits_{x\to1}\left(\dfrac{48x-42}{48x-30}\right)=\dfrac{1}{3}$

Type 3 - L'Hopital's Rule for Transcendental Functions and Trig

5. $\lim\limits_{x\to1}\left(\dfrac{\ln x-x+1}{e^x-e}\right)=\lim\limits_{x\to1}\left(\dfrac{\frac{1}{x}-1}{e^x}\right)=0$

6. $\lim\limits_{x\to0}\left(\dfrac{x^2\sin x+\cos x-1}{x}\right)$

$\lim\limits_{x\to0}\left(\dfrac{x^2\cos x+2x\sin x-\cos x}{1}\right)=0$

Type 4 - L'Hopital's Rule for $\pm\dfrac{\infty}{\infty}$

1. $\lim\limits_{x\to\infty}\dfrac{2x^3-4x^2-5x-2}{3x^3-12}=\lim\limits_{x\to\infty}\dfrac{6x^2-8x-5}{9x^2}$

$\lim\limits_{x\to\infty}\dfrac{12x-18}{18x}=\dfrac{2}{3}$

2. $\lim\limits_{x\to\infty}\dfrac{4x^2-5x+2}{e^{5x}+\ln x}=\lim\limits_{x\to\infty}\dfrac{8x-5}{5e^{5x}+\frac{1}{x}}$

$\lim\limits_{x\to\infty}\dfrac{8}{25e^{5x}-\frac{1}{x^2}}=0$

L'Hopital's Rule for Indeterminate Forms - Homework

Basic Problems - calculate your answers and check on your calculators

1. $\lim_{x\to -1}\frac{2x^2-x-3}{x+1}=\lim_{x\to -1}\frac{4x-1}{1}=-5$

2. $\lim_{x\to 3}\frac{x^2-x-6}{x-3}=\lim_{x\to 3}\frac{2x-1}{1}=5$

3. $\lim_{x\to 3}\frac{x^2-x-3}{x-2}=3$

4. $\lim_{x\to 1}\frac{x-1}{\sqrt{x^2+3}-2}=\lim_{x\to 1}\frac{\sqrt{x^2+3}}{x}=2$

5. $\lim_{x\to 0}\frac{x}{1-e^x}=\lim_{x\to 0}\frac{1}{-e^x}=-1$

6. $\lim_{x\to 1}\frac{x^3-3x+2}{x^3-x^2-x+1}=\lim_{x\to 1}\frac{3x^2-3}{3x^2-2x-1}$
$\lim_{x\to 1}\frac{6x}{6x-2}=\frac{3}{2}$

7. $\lim_{x\to 1}\frac{1-x+\ln x}{x^3-3x+2}=\lim_{x\to 1}\frac{-1+\frac{1}{x}}{3x^2-3}$
$\lim_{x\to 1}\frac{-\frac{1}{x^2}}{6x}=\frac{-1}{6}$

8. $\lim_{x\to 0}\frac{x\cos x-\sin x}{x}=\lim_{x\to 0}\frac{-x\sin x+\cos x-\cos x}{1}=0$

9. $\lim_{x\to\infty}\frac{x^2-1}{4x^2+x}=\lim_{x\to\infty}\frac{2x}{8x+1}=\lim_{x\to\infty}\frac{2}{8}=\frac{1}{4}$

10. $\lim_{x\to\infty}\frac{2x^2+4x-7}{x^3+3x^2-5}=\lim_{x\to\infty}\frac{4x+4}{3x^2+6x}$
$\lim_{x\to\infty}\frac{4}{6x+6}=0$

11. $\lim_{x\to\infty}\frac{x^3}{e^x}=\lim_{x\to\infty}\frac{3x^2}{e^x}=\lim_{x\to\infty}\frac{6x}{e^x}$
$\lim_{x\to\infty}\frac{6}{e^x}=0$

12. $\lim_{x\to\infty}\frac{\sqrt{x^2-1}}{2x+1}=\lim_{x\to\infty}\frac{2x}{2\sqrt{x^2-1}}=\lim_{x\to\infty}\frac{\sqrt{x^2-1}}{2x}$
L'Hopital's Rule doesn't work, but limit $=\frac{1}{2}$