Related Rates - Classwork

Earlier in the year, we used the basic definition of calculus as "the mathematics of change." We defined words that meant change: increasing, decreasing, growing, shrinking, etc. Change occurs over time. So, when we talk about how a quantity changes, we are talking about the derivative of that quantity with respect to time.

Example 1) Write the following statements mathematically.

a) John is growing at the rate of 3 inches/year.

$$\boxed{\frac{dh}{dt} = 3 \text{ in/yr}}$$

b) My mutual fund is shrinking by 4 cents/day.

$$\boxed{\frac{dm}{dt} = -.04 \text{ /day}}$$

c) The radius of a circle is increasing by 4 ft/hr.

$$\boxed{\frac{dr}{dt} = 4 \text{ ft/hr}}$$

d) The volume of a cone is decreasing by 2 in^3/sec.

$$\boxed{\frac{dV}{dt} = -2 \text{ in}^3\text{/sec}}$$

Example 2) A rectangle is 10 inches by 6 inches whose sides are changing. Write formulas for both the perimeter and area and how fast each is changing in terms of L and W.

Perimeter $\boxed{P = 2l + 2w}$

Change of perimeter $\boxed{\frac{dP}{dt} = 2\frac{dl}{dt} + 2\frac{dw}{dt}}$

Area $\boxed{A = lw}$

Change of area $\boxed{\frac{dA}{dt} = l\frac{dw}{dt} + w\frac{dl}{dt}}$

a. its length and width are increasing at the rate rate of 2 inches/sec.

$$\boxed{\frac{dl}{dt} = 2 \text{ in/sec} \quad \frac{dw}{dt} = 2 \text{ in/sec}}$$

Change of perimeter $\boxed{2(2) + 2(2) = 8 \text{ in/sec}}$

Change of area $\boxed{10(2) + 6(2) = 32 \text{ in/sec}}$

b. its length and width are decreasing at the rate of 2 inches/sec.

$$\boxed{\frac{dl}{dt} = -2 \text{ in/sec} \quad \frac{dw}{dt} = -2 \text{ in/sec}}$$

Change of perimeter $\boxed{2(-2) + 2(-2) = -8 \text{ in/sec}}$

Change of area $\boxed{10(-2) + 6(-2) = -32 \text{ in/sec}}$

c. its length is increasing at 3 inches/sec and the width is decreasing at 3 inches/sec.

$$\boxed{\frac{dl}{dt} = 3 \text{ in/sec} \quad \frac{dw}{dt} = -3 \text{ in/sec}}$$

Change of perimeter $\boxed{2(3) + 2(-3) = 0 \ \frac{\text{in}}{\text{sec}}}$

Change of area $\boxed{10(-3) + 6(3) = -12 \ \frac{\text{in}^2}{\text{sec}}}$

d. its length is decreasing at the rate of 2 inches/sec and its width is increasing at .5 inches/sec.

$$\boxed{\frac{dl}{dt} = 2 \text{ in/sec} \quad \frac{dw}{dt} = -.5 \text{ in/sec}}$$

Change of perimeter $\boxed{2(2) + 2(-.5) = 3 \ \frac{\text{in}}{\text{sec}}}$

Change of area $\boxed{10(-.5) + 6(2) = 7 \ \frac{\text{in}^2}{\text{sec}}}$

Example 3) A right triangle has sides of 30 and 40 inches whose sides are changing. Write formulas for the area of the triangle and the hypotenuse of the triangle and how fast the area and hypotenuse are changing

Area $\boxed{A = \frac{1}{2}xy}$

Change of area $\boxed{\frac{dA}{dt} = \frac{1}{2}\left(x\frac{dy}{dt} + y\frac{dx}{dt}\right)}$

Hypotenuse $\boxed{z = \sqrt{x^2 + y^2}}$

Change of hypotenuse $\boxed{\frac{dz}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{\sqrt{x^2 + y^2}}}$

a) the short side is increasing at 3 in./sec and the long side is increasing at 5 in/sec.

$$\boxed{\frac{dx}{dt} = 3 \text{ in/sec} \quad \frac{dy}{dt} = 5 \text{ in/sec}}$$

Change of area $\boxed{\frac{1}{2}\big(30(5) + 40(3)\big) = 135 \ \frac{\text{in}^2}{\text{sec}}}$

Change of hypotenuse $\boxed{\frac{30(3) + 40(5)}{50} = 5.8 \ \frac{\text{in}}{\text{sec}}}$

b) the short side is increasing at 3 in./sec and the long side is decreasing at 5 in./sec.

$$\boxed{\frac{dx}{dt} = 3 \text{ in/sec} \quad \frac{dy}{dt} = -5 \text{ in/sec}}$$

Change of area $\boxed{\frac{1}{2}\big(30(-5) + 40(3)\big) = -15 \ \frac{\text{in}^2}{\text{sec}}}$

Change of hypotenuse $\boxed{\frac{30(3) + 40(-5)}{50} = -2.2 \ \frac{\text{in}}{\text{sec}}}$

Example 4. A right circular cylinder has a height of 10 feet and radius 8 feet whose dimensions are changing. Write formulas for the volume and surface area of the cylinder and the rate at which they change.

Volume	*Change of volume*	*Surface area*	*Change of surface area*
$\boxed{V = \pi r^2 h}$	$\boxed{\dfrac{dV}{dt} = \pi\left(r^2\dfrac{dh}{dt} + 2hr\dfrac{dr}{dt}\right)}$	$\boxed{S = 2\pi rh}$	$\boxed{\dfrac{dS}{dt} = 2\pi\left(r\dfrac{dh}{dt} + h\dfrac{dr}{dt}\right)}$

a) the radius is growing at 2 feet/min and the height is shrinking at 3 feet/min.

$$\boxed{\frac{dr}{dt} = 2 \text{ ft/min} \qquad \frac{dh}{dt} = -3 \text{ ft/min}}$$

Change of volume

$$\boxed{\pi\big(64(-3) + 160(2)\big) = 128\pi \ \tfrac{\text{ft}^3}{\text{min}}}$$

Change of surface area

$$\boxed{2\pi\big(8(-3) + 10(2)\big) = -8\pi \ \tfrac{\text{ft}^2}{\text{min}}}$$

b) the radius is decreasing at 4 feet/min and the the height is increasing at 2 feet/min.

$$\boxed{\frac{dr}{dt} = -4 \text{ ft/min} \qquad \frac{dh}{dt} = 2 \text{ ft/min}}$$

Change of volume

$$\boxed{\pi\big(64(2) + 160(-4)\big) = -512\pi \ \tfrac{\text{ft}^3}{\text{min}}}$$

Change of surface area

$$\boxed{2\pi\big(8(2) + 10(-4)\big) = -48\pi \ \tfrac{\text{ft}^2}{\text{min}}}$$

To solve related rates problems, you need a strategy that always works. Related rates problems always can be recognized by the words "increasing, decreasing, growing, shrinking, changing." Follow these guidelines in solving a related rates problem.

1. Make a sketch. Label all sides in terms of variables even if you are given the actual values of the sides.

2. You will make a table of variables. The table will contain two types of variables - variables that are constants and variables that are changing. Variables that never change go into the constant column. Variables that are a given value only at a certain point in time go into the changing column. Rates (recognized by "increasing", "decreasing", etc.) are derivatives with respect to time and can go in either column.

3. Find an equation which ties your variables together. If it an area problem, you need an area equation. If it is a right triangle, the Pythagorean formula may work or gerenal trig formulas may apply. If it is a general triangle, the law of cosines may work.

4. You may now plug in any variable in the constant column. **Never** plug in any variable in the changing column.

5. Differentiate your equation with respect to time. You are doing implicit differentiation with respect to *t*.

6. Plug in all variables. Hopefully, you will know all variables except one. If not, you will need an equation which will solve for unknown variables. Many times, it is the same equation as the one you used above. Do this work on the side as to not destroy the momentum of your work so far.

7. Label your answers in terms of the correct units (very important) and be sure you answered the question asked.

Example 5) An oil tanker spills oil that spreads in a circular pattern whose radius increases at the rate of 50 feet/min. How fast are both the circumference and area of the spill increasing when the radius of the spill is a) 20 feet and b) 50 feet?

$$C = 2\pi r$$
$$\frac{dC}{dt} = 2\pi\frac{dr}{dt}$$
$$\text{a,b}: \ \frac{dC}{dt} = 2\pi(50) = 100\pi \ \frac{\text{ft}}{\text{min}}$$

$$A = \pi r^2$$
$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$
$$\text{a)}\ \frac{dA}{dt} = 2\pi(20)(50) = 2000\pi \ \frac{\text{ft}^2}{\text{min}}$$

$$A = \pi r^2$$
$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$
$$\text{b)}\ \frac{dA}{dt} = 2\pi(50)(50) = 5000\pi \ \frac{\text{ft}^2}{\text{min}}$$

Example 6) A 13 foot ladder leans against a vertical wall. If the lower end of the ladder is pulled away at the rate 2 feet per second, how fast is the top of the ladder coming down the wall at a) the instant the top is 12 feet above the ground and b) 5 feet above the ground?

$x^2+y^2=169 \qquad x^2+144=169 \Rightarrow x=5$	$x^2+y^2=169 \qquad x^2+25=169 \Rightarrow x=12$
$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$	$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$
$2(5)(2)+2(12)\frac{dy}{dt}=0$	$2(12)(2)+2(5)\frac{dy}{dt}=0$
$\frac{dy}{dt}=\frac{6}{5}=1.20 \text{ ft/sec}$	$\frac{dy}{dt}=\frac{-24}{5}=-4.8 \text{ ft/sec}$

Example 7) A camera is mounted 3,000 feet from the space shuttle launching pad. The camera needs to pivot as the shuttle is launched and needs to keep the shuttle in focus. If the shuttle is rising vertically at 800 feet/sec when it is 4,000 feet high, how fast is the camera-to-shuttle distance changing?

$$x^2+y^2=z^2 \qquad 3000^2+4000^4=z^2 \Rightarrow z=5000$$
$$3000^2+y^2=z^2$$
$$2y\frac{dy}{dt}=2z\frac{dz}{dt}$$
$$2(4000)(800)=2(5000)\frac{dz}{dt}$$
$$\frac{dz}{dt}=640\ \frac{\text{ft}}{\text{sec}}$$

In this problem, how fast is the angle of elevation of the camera changing at that moment in time?

What variable are we trying to find? $\boxed{\frac{d\theta}{dt}}$ Since this is a function of θ, we need a trig function.

There are three trig functions we could use. Let's try all three and determine which is best.

$\sin\theta=\frac{y}{z}$	$\cos\theta=\frac{x}{z}$	$\tan\theta=\frac{y}{x}$
$\cos\theta\frac{d\theta}{dt}=\frac{z\frac{dy}{dt}-y\frac{dz}{dt}}{z^2}$	$-\sin\theta\frac{d\theta}{dt}=\frac{-x\frac{dz}{dt}}{z^2}$	$\sec^2\theta\frac{d\theta}{dt}=\frac{1}{3000}\frac{dy}{dt}$
$\frac{3}{5}\frac{d\theta}{dt}=\frac{5000(800)-4000(640)}{5000^2}$	$\frac{-4}{5}\frac{d\theta}{dt}=\frac{-3000(640)}{5000^2}$	$\left(\frac{5}{3}\right)^2\frac{d\theta}{dt}=\frac{1}{3000}(800)$
$\frac{d\theta}{dt}=.096\ \frac{\text{R}}{\text{sec}}$	$\frac{d\theta}{dt}=.096\ \frac{\text{R}}{\text{sec}}$	$\frac{d\theta}{dt}=.096\ \frac{\text{R}}{\text{sec}}$

$$\boxed{\frac{d\theta}{dt}\approx 5.5^o \text{ sec}}$$

Example 8) Two cars are riding on roads that meet at a $60°$ angle. Car A is 3 miles from the intersection traveling at 40 mph and car B is 2 miles away from the intersection traveling at 50 mph. How fast are the two cars separating if a) they are both traveling away from the intersection and b) car A is traveling away from the intersection and car B is traveling towards it?

$$c^2 = a^2 + b^2 - 2ab\cos 60^o$$

$$c^2 = a^2 + b^2 - ab \Rightarrow c^2 = 9 + 4 - 6 \Rightarrow c = \sqrt{7}$$

$$2c\frac{dc}{dt} = 2a\frac{da}{dt} + 2b\frac{db}{dt} - a\frac{db}{dt} - b\frac{da}{dt}$$

$$2\sqrt{7}\frac{dc}{dt} = 2(3)(40) + 2(2)(50) - 3(50) - 2(40)$$

$$\frac{dc}{dt} = \frac{210}{2\sqrt{7}} = 39.686 \text{ mph}$$

$$2c\frac{dc}{dt} = 2a\frac{da}{dt} + 2b\frac{db}{dt} - a\frac{db}{dt} - b\frac{da}{dt}$$

$$2\sqrt{7}\frac{dc}{dt} = 2(3)(40) + 2(2)(-50) - 3(-50) - 2(40)$$

$$\frac{dc}{dt} = \frac{110}{2\sqrt{7}} = 20.788 \text{ mph}$$

Example 9) Sand is poured on a beach creating a cone whose radius is always equal to twice its height. If the sand is poured at the rate of $20 \text{ in}^3/\text{sec}$ How fast is the height changing at the time the height is a) 2 inches, b) 6 inches?

$$V = \frac{1}{3}\pi r^2 h \qquad r = 2h$$

$$V = \frac{1}{3}\pi(2h)^2 h$$

$$V = \frac{4}{3}\pi h^3$$

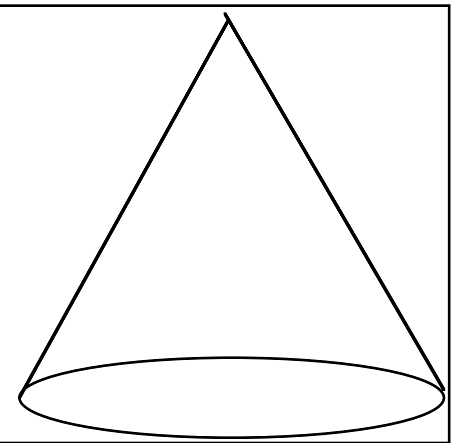
$$\frac{dV}{dt} = 4\pi h^2 \frac{dh}{dt}$$

$$20 = 4\pi(2^2)\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{20}{16\pi} = .398 \frac{\text{in}}{\text{sec}}$$

$$20 = 4\pi(6^2)\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{20}{144\pi} = .044 \frac{\text{in}}{\text{sec}}$$



Example 10) Water is draining from a conical tank at the rate of $2 \text{ meter}^3/\text{sec}$. The tank is 16 meters high and its top radius is 4 meters. How fast is the water level falling when the water level is a) 12 meters high, b) 2 meters high?

$$V = \frac{1}{3}\pi r^2 h \qquad \frac{r}{h} = \frac{4}{16} \Rightarrow r = \frac{1}{4}h$$

$$V = \frac{1}{3}\pi\left(\frac{1}{4}h\right)^2 h$$

$$V = \frac{1}{48}\pi h^3$$

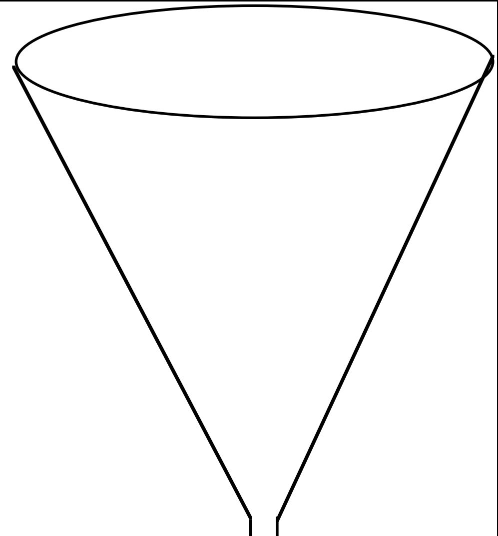
$$\frac{dV}{dt} = \frac{1}{16}\pi h^2 \frac{dh}{dt}$$

$$2 = \frac{1}{16}\pi(12^2)\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{32}{144\pi} = .071 \frac{\text{m}}{\text{min}}$$

$$2 = \frac{1}{16}\pi(2^2)\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{32}{4\pi} = 2.546 \frac{\text{m}}{\text{min}}$$



Related Rates - Homework

1. A circle has a radius of 8 inches which is changing. Write formulas for its circumference and area.

Circumference $\boxed{C = 2\pi r}$ *Change of circumference* $\boxed{\dfrac{dC}{dt} = 2\pi\dfrac{dr}{dt}}$ *Area* $\boxed{A = \pi r^2}$ *Change of area* $\boxed{\dfrac{dA}{dt} = 2\pi r\dfrac{dr}{dt}}$

a) its radius is growing at the rate of 3 in./min.

$\boxed{\dfrac{dr}{dt} = 3\text{ in/min}}$

Change of cirumference $\boxed{\dfrac{dC}{dt} = 6\pi\text{ in/min}}$ *Change of area* $\boxed{\dfrac{dA}{dt} = 48\pi\text{ in}^2\text{/min}}$

b) its radius is shrinking at the rate of $\frac{1}{4}$ inch/sec.

$\boxed{\dfrac{dr}{dt} = -\dfrac{1}{4}\text{ in/sec}}$

Change of cirumference $\boxed{\dfrac{dC}{dt} = -\dfrac{1}{2}\text{ in/sec}}$ *Change of area* $\boxed{\dfrac{dA}{dt} = -4\pi\text{ in}^2\text{/sec}}$

c) its diameter is growing at the rate of 4 in/min.

$\boxed{\dfrac{dr}{dt} = 2\text{ in/min}}$

Change of cirumference $\boxed{\dfrac{dC}{dt} = 4\pi\text{ in/min}}$ *Change of area* $\boxed{\dfrac{dC}{dt} = 32\pi\text{ in}^2\text{/min}}$

d) its radius is shrinking at the rate of 1 inch/sec.

$\boxed{\dfrac{dr}{dt} = -1\text{ in/sec}}$

Change of cirumference $\boxed{\dfrac{dC}{dt} = -2\pi\text{ in/sec}}$ *Change of area* $\boxed{\dfrac{dA}{dt} = -16\pi\text{ in}^2\text{/sec}}$

2. A sphere has a radius of 9 feet which is changing. Write formulas for its volume and surface area.

Volume $\boxed{V = \dfrac{4}{3}\pi r^3}$ *Change of volume* $\boxed{\dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt}}$ *Surface area* $\boxed{S = 4\pi r^2}$ *Change of surface area* $\boxed{\dfrac{dS}{dt} = 8\pi r\dfrac{dr}{dt}}$

a) its diameter is growing at the rate of 1 ft/min.

$\boxed{\dfrac{dr}{dt} = \dfrac{1}{2}\dfrac{\text{ft}}{\text{min}}}$

Change of volume $\boxed{\dfrac{dV}{dt} = 162\pi\dfrac{\text{ft}^3}{\text{min}}}$ *Change of surface area* $\boxed{\dfrac{dS}{dt} = 36\pi\dfrac{\text{ft}^2}{\text{min}}}$

b) its radius is shrinking at the rate of 9 inch/sec.

$\boxed{\dfrac{dr}{dt} = \dfrac{-3}{4}\dfrac{\text{ft}}{\text{sec}}}$

Change of volume $\boxed{\dfrac{dV}{dt} = -243\pi\dfrac{\text{ft}^3}{\text{sec}}}$ *Change of surface area* $\boxed{\dfrac{dS}{dt} = -54\pi\dfrac{\text{ft}^2}{\text{sec}}}$

3. A right circular cone has a height of 10 feet and radius 6 feet, both of which are changing. Write a formula for the volume of the cone.

Volume $\boxed{V = \dfrac{1}{3}\pi r^2 h}$ *Change of volume* $\boxed{\dfrac{dV}{dt} = \dfrac{1}{3}\pi\left(r^2\dfrac{dh}{dt} + 2rh\dfrac{dr}{dt}\right)}$

a) the radius grows at 6 ft/sec, the height shrinks at 1 ft/sec

Change of volume $\boxed{\dfrac{dV}{dt} = 228\pi\dfrac{\text{ft}^3}{\text{sec}}}$

b) the diameter is shrinking at 6 ft/sec and the height is growing at 2 ft/sec.

Change of volume $\boxed{\dfrac{dV}{dt} = -96\pi\dfrac{\text{ft}^3}{\text{sec}}}$

4. A rectangular well is 6 feet long, 4 feet wide, and 8 feet deep. If water is running into the well at the rate of 3 ft^3/sec, find how fast the water is rising (keep in mind which variables are constant and which are changing).

$$\boxed{\begin{array}{l} V = lwh \\ V = 24h \end{array}} \qquad \boxed{\begin{array}{l} 3 = 24\dfrac{dh}{dt} \\ \dfrac{dh}{dt} = \dfrac{1}{8} \text{ ft/sec} \end{array}}$$

5. A spherical hot air balloon is being inflated. If air is blown into the ballon at the rate of 2 ft^3/sec,

a. find how fast the radius of the balloon is changing when the radius is 3 feet.

b. find how fast the surface area is increasing at the same time.

$$\boxed{\begin{array}{l} V = \dfrac{4}{3}\pi r^3 \\ \dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt} \end{array}} \qquad \boxed{\begin{array}{l} 2 = 36\pi \dfrac{dr}{dt} \\ \dfrac{dr}{dt} = \dfrac{1}{18\pi} \text{ ft/sec} \end{array}}$$

$$\boxed{\begin{array}{l} S = 4\pi r^2 \\ \dfrac{dS}{dt} = 8\pi r \dfrac{dr}{dt} \end{array}} \qquad \boxed{\begin{array}{l} \dfrac{dS}{dt} = 24\pi\left(\dfrac{1}{18\pi}\right) \\ \dfrac{dr}{dt} = \dfrac{4}{3} \text{ ft}^2\text{/sec} \end{array}}$$

6. A 12 foot ladder stands against a vertical wall. If the lower end of the ladder is being pulled away from the wall at the rate of 2 ft/sec,

a) how fast is the top of the ladder coming down the wall at the instant it is 6 feet above the ground?

b. how fast is the angle of the elevation of the ladder changing at the same instant?

$$\boxed{\begin{array}{l} x^2 + y^2 = 144 \qquad x^2 + 36 = 144 \Rightarrow x = \sqrt{108} \\ 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0 \\ 2\sqrt{108}(2) + 2(6)\dfrac{dy}{dt} = 0 \\ \dfrac{dy}{dt} = \dfrac{-\sqrt{108}}{3} = -3.464 \text{ ft/sec} \end{array}}$$

$$\boxed{\begin{array}{l} \cos\theta = \dfrac{x}{12} \\ -\sin\theta \dfrac{d\theta}{dt} = \dfrac{1}{12}\dfrac{dx}{dt} \\ -\dfrac{6}{12}\dfrac{d\theta}{dt} = \dfrac{1}{12}(2) \\ \dfrac{d\theta}{dt} = \dfrac{-1}{3} \text{ R/sec} \approx \text{-}19.1°\text{/sec} \end{array}}$$

7. Superman is in level flight 6 miles above ground. His flight plan takes him directly over Wissahickon High. How fast is he flying when the distance between him and WHS is exactly 10 miles and this distance is increasing at the rate of 40 mph?

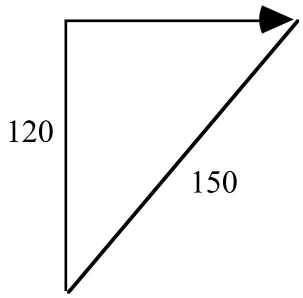
$$\boxed{\begin{array}{l} x^2 + y^2 = r^2 \qquad x^2 + 36 = 100 \Rightarrow x = 8 \\ x\dfrac{dx}{dt} = r\dfrac{dr}{dt} \end{array}} \qquad \boxed{\begin{array}{l} 8\dfrac{dx}{dt} = 10(40) \\ \dfrac{dx}{dt} = 50 \text{ mph} \end{array}}$$

8. Two roads meet at an angle of 60°. A man starts from the intersection at 1 PM and walks along one road at 3 mph. At 2:00 PM, another man starts along the second road and walks at 4 mph. How fast are they separating at 4 PM?

$$\boxed{\begin{array}{l} c^2 = a^2 + b^2 - 2ab\cos 60^o \\ c^2 = a^2 + b^2 - ab \Rightarrow c^2 = 81 + 64 - 72 \Rightarrow c = \sqrt{73} \end{array}}$$

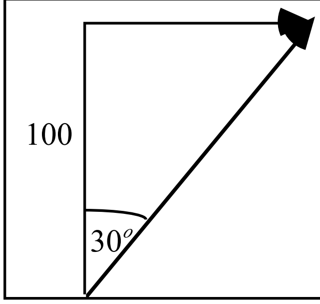
$$\boxed{\begin{array}{l} 2c\dfrac{dc}{dt} = 2a\dfrac{da}{dt} + 2b\dfrac{db}{dt} - a\dfrac{db}{dt} - b\dfrac{da}{dt} \\ 2\sqrt{73}\dfrac{dc}{dt} = 2(9)(3) + 2(8)(4) - 9(4) - 8(3) \\ \dfrac{dc}{dt} = 3.394 \text{ mph} \end{array}}$$

9. A boy flies a kite which is 120 ft directly above his hand. If the wind carries the kite horizontally at the rate of 30 ft/min, at what rate is the string being pulled out when the length of the string is 150 ft?



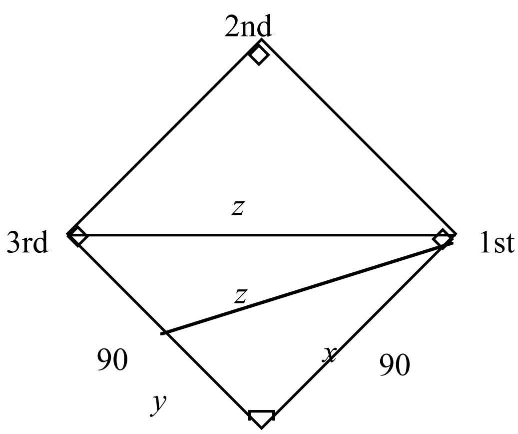
$$x^2 + y^2 = z^2$$
$$x\frac{dx}{dt} = z\frac{dz}{dt}$$
$$90(30) = 150z\frac{dz}{dt}$$
$$\frac{dz}{dt} = 18 \text{ ft/min}$$

10. The same boy flies a kite which is now 100 feet above the ground. If the string is pulled out at the rate of 10 ft/sec because the wind carries the kite horizontally directly away from the boy, what is the rate of change of the angle the kite makes with the vertical when the angle is 30º.



$$\cos\theta = \frac{y}{z}$$
$$-\sin\theta\frac{d\theta}{dt} = \frac{-100}{z^2}\frac{dz}{dt}$$
$$\frac{1}{2}\frac{d\theta}{dt} = \frac{100}{\left(\frac{100}{\sqrt{3}}\right)^2}(10)$$
$$\frac{d\theta}{dt} = .15R/\sec \approx 8.6^o/\sec$$

11. A baseball diamond is a 90-foot square. A ball is batted along the third-base line at a constant rate of 100 feet per second. How fast is its distance changing from first base at the time when a) the ball is halfway to 3rd base and b) it reaches 3rd base.



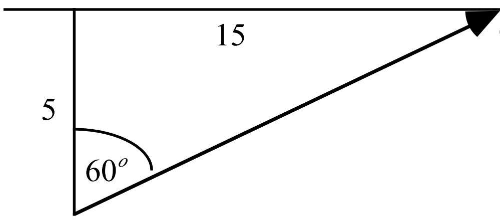
$$y\frac{dy}{dt} = z\frac{dz}{dt}$$
$$\frac{dz}{dt} = \frac{45(100)}{\sqrt{10125}} \qquad \frac{dz}{dt} = \frac{90(100)}{\sqrt{16200}}$$
$$\frac{dz}{dt} = 44.721 \text{ ft/sec} \qquad \frac{dz}{dt} = 70.711 \text{ ft/sec}$$

12. A plane is flying west at 500 ft/sec at an altitude of 4,000 ft. The plane is tracked by a searchlight on the ground. If the light is to be trained on the plane, find the change in the angle of elevation of the searchlight at a horizontal distance of 2,000 ft.

$$\tan\theta = \frac{x}{y}$$
$$\sec^2\theta\frac{d\theta}{dt} = \frac{1}{4000}\frac{dx}{dt}$$
$$\left(\frac{\sqrt{20}}{4}\right)^2\frac{d\theta}{dt} = \frac{1}{4000}(500)$$

$$\frac{5}{4}\frac{d\theta}{dt} = \frac{1}{8}$$
$$\frac{d\theta}{dt} = \frac{1}{10}R/\sec \approx 5.730^o/\sec$$

13. A revolving light located 5 miles from a straight shore line turns with a constant angular velocity. What velocity does the light revolve if the light moves along the shore at the rate of 15 miles per minute when the beam makes an angle of 60º with the shore line?



Given: $y = 5\ (\text{const})$

$\frac{dx}{dt} = 15$

$\theta = 60^o$

$\frac{d\theta}{dt} = ?$

$\tan\theta = \frac{x}{y}$

$\tan\theta = \frac{x}{5}$

$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{5}\frac{dx}{dt}$

$\sec^2 60^o \frac{d\theta}{dt} = \frac{1}{5}(15)$

$2^2\frac{d\theta}{dt} = 3$

$\frac{d\theta}{dt} = \frac{3}{4}\frac{radian}{\text{min}} \approx \frac{42.8^o}{\text{min}}$

14. How fast does the radius of a spherical soap bubble change when you blow air into it at the rate of 10 cm^3/sec at the time when the radius is 2 cm?

Given: $\frac{dV}{dt} = 10$

$r = 2$

$\frac{dr}{dt} = ?$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$10 = 4\pi(4)\frac{dr}{dt}$

$\frac{dr}{dt} = \frac{5}{8\pi} = .199\frac{\text{cm}}{\text{sec}}$

15. How fast does the water level drop when a cylindrical tank of radius 6 feet is drained at the rate of 3 ft^3/min?

Given: $r = 6\ (\text{const})$

$\frac{dV}{dt} = 3$

$\frac{dh}{dt} = ?$

$V = \pi r^2 h$

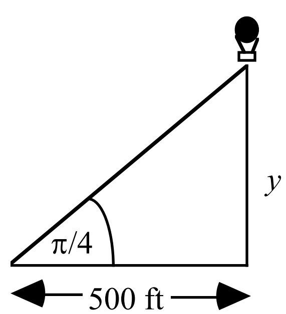
$V = 36\pi h$

$\frac{dV}{dt} = 36,,\frac{dh}{dt}$

$3 = 36\pi\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{1}{12\pi} = .027\frac{\text{ft}}{\text{min}}$

16. A hot air balloon, rising straight up from a level field, is tracked by a range finder 500 feet from the lift-off point. At the moment the range finder's elevation angle is $\frac{\pi}{4}$, the angle is increasing at the rate of 0.14 radians/min. How fast is the balloon rising?



Given

$x = 500\ (\text{const})$

$\theta = \frac{\pi}{4}$

$\frac{d\theta}{dt} = \frac{.14^r}{\text{min}}$

$\frac{dy}{dt} = ?$

$\tan\theta = \frac{y}{x}$

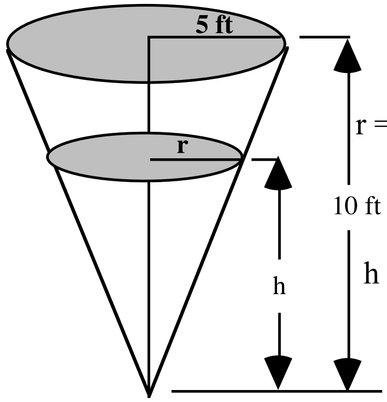
$\tan\theta = \frac{y}{500}$

$\sec^2\theta\frac{d\theta}{dt} = \frac{1}{500}\frac{dy}{dt}$

$2(.14) = \frac{1}{500}\frac{dy}{dt}$

$\frac{dy}{dt} = 140\frac{\text{ft}}{\text{min}}$

17. Water runs into a conical tank at the rate of 9 ft^3/min. The tank stands vertex down and has a height of 10 feet and a base radius of 5 feet. How fast is the water level rising when the water is 6 feet deep?



r = radius of water surface at time t

h = depth of water surface at time t

Given

$r = 5$

$h = 6$

$\dfrac{dV}{dt} = 9\dfrac{ft^3}{\text{min}}$

$\dfrac{dh}{dt} = ?$

$V = \dfrac{1}{3}\pi r^2 h$

$V = \dfrac{1}{3}\pi\left(\dfrac{h}{2}\right)^2 h$

$V = \dfrac{1}{12}\pi h^3$

$\dfrac{dV}{dt} = \dfrac{\pi}{4}h^2\dfrac{dh}{dt}$

$9 = \dfrac{\pi}{4}6^2\dfrac{dh}{dt}$

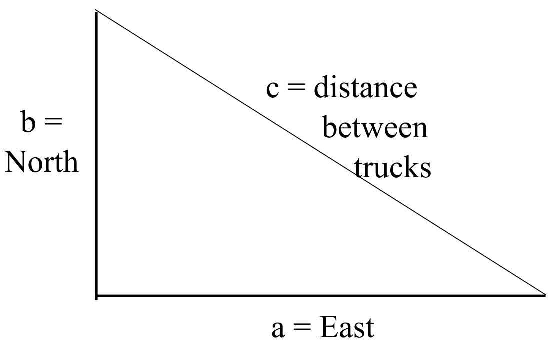
$\dfrac{dh}{dt} = \dfrac{1}{\pi} = .318\dfrac{\text{ft}}{\text{min}}$

By similar triangles

$\dfrac{5}{10} = \dfrac{r}{h}$

$r = \dfrac{h}{2}$

18. Two truck convoys leave a depot at the same time. Convoy A travels east at 40 mph and convoy B travels north at 30 mph. How fast is the distance between the convoys changing a) in 6 minutes b) in 30 minutes



Given

$\dfrac{da}{dt} = 40 \text{ mph}$

$\dfrac{db}{dt} = 30 \text{ mph}$

$a^2 + b^2 = c^2$

$2a\dfrac{da}{dt} + 2b\dfrac{db}{dt} = 2c\dfrac{dc}{dt}$

Side Work

$a^2 + b^2 = c^2$

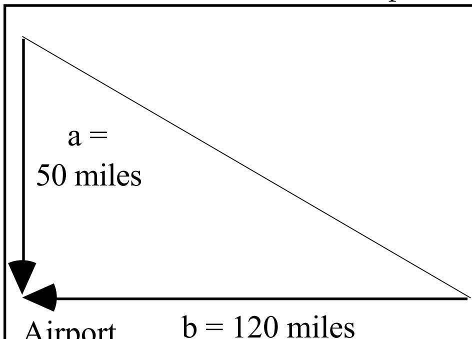
$a = \dfrac{1}{10}(40) = 4 \text{ miles}$ $\qquad 4(40) + 3(30) = 5\dfrac{dc}{dt}$ $\qquad c = 5$

$b = \dfrac{1}{10}(30) = 3 \text{ miles}$ $\qquad \boxed{\dfrac{dc}{dt} = 50 \text{ mph}}$

$a = \dfrac{1}{2}(40) = 20 \text{ miles}$ $\qquad 20(40) + 15(30) = 25\dfrac{dc}{dt}$ $\qquad a^2 + b^2 = c^2$

$b = \dfrac{1}{2}(30) = 15 \text{ miles}$ $\qquad \boxed{\dfrac{dc}{dt} = 50 \text{ mph}}$ $\qquad c = 25$

19. Two commercial jets at 40,000 ft. are both flying at 520 mph towards an airport. Plane A is flying south and is 50 miles from the airport while Plane B is flying west and is 120 miles from the airport. How fast is the distance between the two planes changing at this time?



Given

$a = 50 \text{ miles}$

$b = 120 \text{ miles}$

$\dfrac{da}{dt} = 520 \text{ mph}$

$\dfrac{db}{dt} = 520 \text{ mph}$

$c^2 = a^2 + b^2$

$2c\dfrac{dc}{dt} = 2a\dfrac{da}{dt} + 2b\dfrac{db}{dt}$

$130\dfrac{dc}{dt} = 50(520) + 120(520)$

$130\dfrac{dc}{dt} = 88400$

$\boxed{\dfrac{dc}{dt} = 680 \text{ mph}}$

Side Work

$c^2 = a^2 + b^2$

$c^2 = 2500 + 14400$

$c = 130$

20. A spherical Tootsie Roll Pop that you are enjoying is giving up volume at a steady rate of 0.25 in^3/min. How fast will the radius be decreasing when the Tootsie Roll Pop is .75 inches across?

Given: $\dfrac{dV}{dt} = -.025$	$V = \dfrac{4}{3}\pi r^3$	$\dfrac{dr}{dt} = -.141\dfrac{\text{in}}{\text{min}}$
$r = .375$	$\dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt}$	
$\dfrac{dr}{dt} = ?$	$-.25 = 4\pi(.375)^2 \dfrac{dr}{dt}$	

21. The mechanics at Toyota Automotive are reboring a 6-inch deep cylinder to fit a new piston. The machine that they are using increases the cylinder's radius one-thousandth of an inch every 3 minutes. How rapidly is the cylinder volume increasing when the bore (diameter) is 3.80 inches?

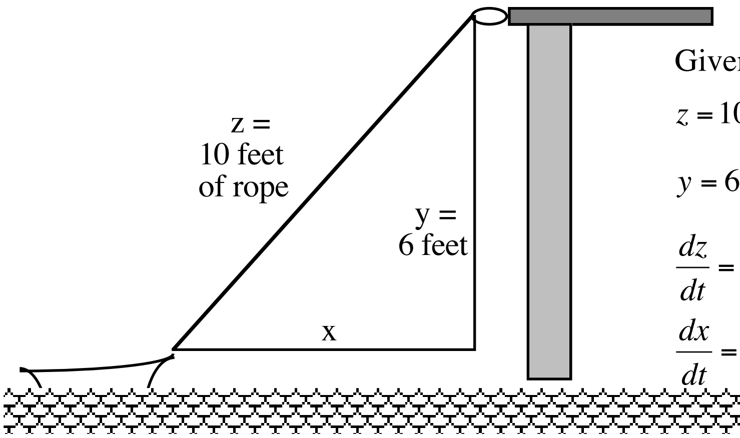
Given: $\dfrac{dr}{dt} = -\dfrac{.001}{3}$	$V = \pi r^2 h$	$\dfrac{dV}{dt} = .024\dfrac{\text{in}^3}{\text{min}}$
$r = 1.9$	$V = 6\pi r^2$	
$h = 6$ (const)	$\dfrac{dV}{dt} = 12\pi r\dfrac{dr}{dt}$	
$\dfrac{dV}{dt} = ?$	$\dfrac{dV}{dt} = 12\pi(1.9)\dfrac{.001}{3}$	

22. Sand falls at the rate of 30 ft^3/min onto the top of a conical pile. The height of the pile is always $\dfrac{3}{8}$ of the base diameter. How fast is the height changing when the pile is 12 ft. high?

Given: $\dfrac{dV}{dt} = 30$	$V = \dfrac{1}{3}\pi r^2 h$	Side Work
$h = 12$	$V = \dfrac{1}{3}\pi\left(\dfrac{16}{9}h^2\right)h$	$h = \dfrac{3}{8}(2r)$
$\dfrac{dh}{dt} = ?$	$\dfrac{dV}{dt} = \pi\left(\dfrac{16}{9}h^2\right)\dfrac{dh}{dt}$	$h = \dfrac{3}{4}r \Rightarrow r = \dfrac{4}{3}h$

$$30 = \frac{16\pi}{9}(144)\frac{dh}{dt}$$
$$\frac{dh}{dt} = \frac{15}{128\pi} = .037\frac{\text{ft}}{\text{min}}$$

23. A rowboat is pulled toward a dock from the bow through a ring on the dock 6 feet above the bow. If the rope is hauled in at 2 ft/sec, how fast is the boat approaching the dock when 10 feet of rope are out?



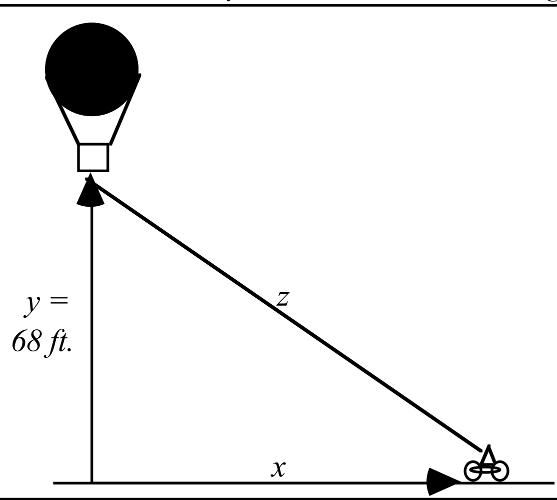
Given		Side Work
$z = 10$	$x^2 + y^2 = z^2$	$x^2 + y^2 = z^2$
$y = 6$ (const.)	$x^2 + 36 = z^2$	$x^2 + 36 = 100$
$\frac{dz}{dt} = 2\frac{\text{ft}}{\text{sec}}$	$2x\frac{dx}{dt} = 2z\frac{dz}{dt}$	$x = 8$
$\frac{dx}{dt} = ?$	$8\frac{dx}{dt} = 10(2)$	
	$\boxed{\frac{dx}{dt} = 2.5\frac{\text{ft}}{\text{sec}}}$	

24. A particle is moving along the curve whose equation is $\frac{xy^3}{1+y^2} = \frac{8}{5}$. Assume the x-coordinate is increasing at the rate of 6 units/sec when the particle is at the point (1, 2). At what rate is the y-coordinate of the point changing at that instant. Is it rising or falling?

Given:	
$x = 1$	$5xy^3 = 8 + 8y^2$
$y = 2$	$5\left(x \cdot 3y^2\frac{dy}{dt} + y^3\frac{dx}{dt}\right) = 16y\frac{dy}{dt}$
$\frac{dx}{dt} = 6$	$5\left(12\frac{dy}{dt} + 8 \cdot 6\right) = 32\frac{dy}{dt}$
$\frac{dy}{dt} = ?$	$28\frac{dy}{dt} = -240$

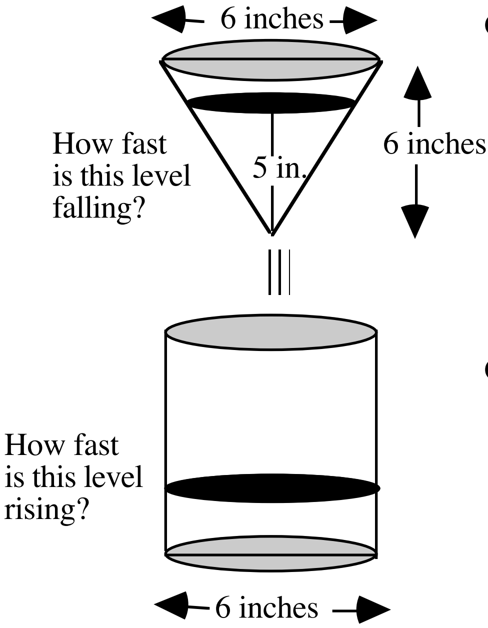
$\boxed{\frac{dy}{dt} = -\frac{60}{7}\frac{\text{units}}{\text{sec}} \text{ - falling}}$

25. A balloon is rising vertically above a level, straight road at a constant rate of 1 foot/sec. Just when the balloon is 65 feet above the ground, a bicycle passes under it going 17 feet/sec. How fast is the distance between the bicycle and balloon increasing 3 seconds later?



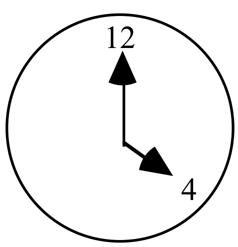
Given		Side Work
$y = 68$ feet	$x^2 + y^2 = z^2$	$x^2 + y^2 = z^2$
$x = 51$ feet	$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$	$51^2 + 65^2 = z^2$
$\frac{dy}{dt} = 1\frac{\text{ft}}{\text{sec}}$	$51(17) + 68(1) = 85\frac{dz}{dt}$	$z = 85$
$\frac{dx}{dt} = 17\frac{\text{ft}}{\text{sec}}$	$\frac{dz}{dt} = \frac{932}{85}$	
$\frac{dz}{dt} = ?$	$\boxed{\frac{dz}{dt} = 11\frac{\text{ft}}{\text{sec}}}$	

26. Coffee is draining from a conical filter into a cylindrical coffeepot at the rate of 10 in^3/min. a) How fast is the level in the pot rising when the coffee in the filter is 5 inches deep? b) How fast is the level in the cone falling then?



Cylinder	Cylinder	Cone	Side Work
$r = 3(\text{const})$	$V = \pi r^2 h$	$V = \frac{1}{3}\pi r^2 h$	$h = 2r$
$\frac{dV}{dt} = 10\frac{\text{in}^3}{\text{min}}$	$V = 9\pi h$	$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$	
$\frac{dh}{dt} = ?$	$\frac{dV}{dt} = 9\pi\frac{dh}{dt}$	$V = \frac{\pi h^3}{12}$	
Cone	$10 = 9\pi\frac{dh}{dt}$	$\frac{dV}{dt} = \frac{\pi h^2}{4}\frac{dh}{dt}$	
$\frac{dV}{dt} = 10\frac{\text{in}^3}{\text{min}}$	$\frac{dh}{dt} = \frac{10}{9\pi}$	$10 = \frac{25\pi}{4}\frac{dh}{dt}$	
$h = 5\ in.$	$\frac{dh}{dt} = \frac{.35\text{ in}}{\text{min}}$	$\frac{dh}{dt} = \frac{8}{5\pi}$	
$\frac{dh}{dt} = ?$		$\frac{dh}{dt} = .51\frac{\text{in}}{\text{min}}$	

27. On a certain clock, the minute hand is 4 in. long and the hour hand is 3 in. long. How fast is the distance between the tips of the hands changing at 4 P.M?



$$c^2 = a^2 + b^2 - 2ab\cos\theta \qquad \frac{d\theta}{dt} = \frac{1\text{ Rev}}{12\text{ hrs}} - \frac{1\text{ Rev}}{\text{hr}} = \frac{2\pi}{12} - 2\pi = -\frac{11\pi}{6}$$

$$c^2 = 16 + 9 - 24\cos\theta \qquad \sqrt{16+9+12} = \sqrt{37}$$

$$2c\frac{dc}{dt} = 24\sin\theta\frac{d\theta}{dt}$$

$$2\sqrt{37}\frac{dc}{dt} = 24\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{11\pi}{6}\right)$$

$$\frac{dc}{dt} = \frac{-11\pi\sqrt{3}}{\sqrt{37}} = -9.84\ \text{in/hr} = -0.164\ \text{in/min}$$

Straight Line Motion - Classwork

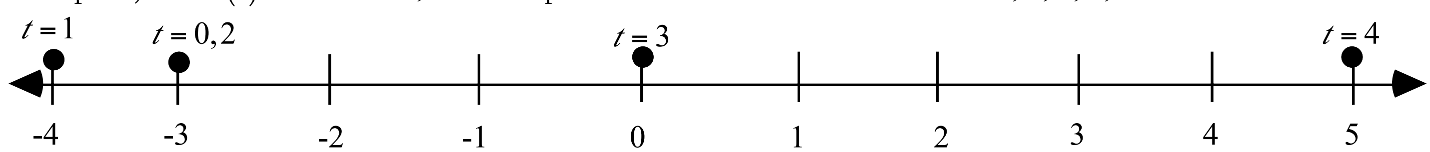
Consider an object moving along a straight line, either horizontally or vertically. There are many such objects, natural, and man-made. Write down several of them.

Horizontal: cars, water

Vertical: rockets, objects subjected to gravity

As an object moves, its position is a function of time. For its position function, we will denote the variable $s(t)$. For instance, when $s(t) = t^2 - 2t - 3$, t in seconds, $s(t)$, we are being told what position on the horizontal or vertical number line the particle occupies at different values of t.

Example 1) For $s(t) = t^2 - 2t - 3$, show its position on the number line for $t = 0, 1, 2, 3, 4$.



When an object moves, its position changes over time. So we can say that the velocity function, $v(t)$ is the change of the position function over time. We know this to be a derivative, and can thus say that $v(t) = s'(t)$.

For convenience sake, we will define $v(t)$ in the following way.

Motion	$v(t) > 0$	$v(t) < 0$	$v(t) = 0$
Horizontal Line	object moves to the right	object moves to the left	object stopped
Vertical Line	object moves up	object moves down	object stopped

Speed is not synonymous with velocity. Speed does not indicate direction. So we define the speed function: $\text{speed} = |v(t)|$. The speed of an object must either be positive or zero (meaning that the object is stopped).

The definition of acceleration is the change of velocity over time. We know this to be a derivative and can thus say that $a(t) = v'(t) = s''(t)$. So given the position function $s(t)$, we can now determine both the velocity and acceleration function. On your cars, you have two devices to change the velocity: accelerator, brake
Let us think as something accelerating the object to be some external force like wind or current. For convenience sake, let us define the acceleration function like this:

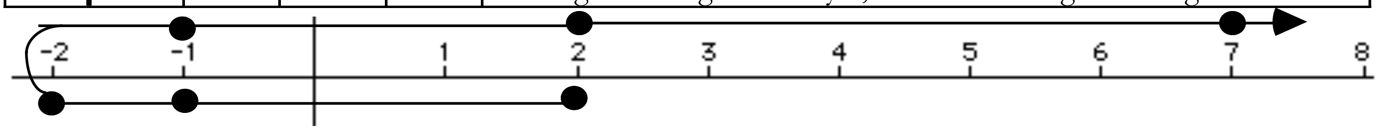
Motion	$a(t) > 0$	$a(t) < 0$	$a(t) = 0$
Horizontal Line	object accelerating to the right	object accelerating to the left	velocity not changing
Vertical Line	object accelerating upwards	object accelerating downwards	velocity not changing

Just because an object's acceleration is zero does not mean that the object is stopped. It means that the velocity is not changing. What device do you have on your cars that keeps the car's acceleration equal to zero? cruise control

Also, just because you have a positive acceleration does not mean that you are moving to the right. For instance, suppose you were walking to the right $[v(t) > 0]$, when all of a sudden a large wind started to blow to the left $[a(t) < 0]$. What would that do to your velocity? slow you down.

Example 2) Given that a particle is moving along a horizontal line with position function $s(t) = t^2 - 4t + 2$.
The velocity function $v(t) = 2t - 4$ and the acceleration function $a(t) = 2$.
Let's complete the chart for the first 5 seconds and show where the object is on the number line.

t	$s(t)$	$v(t)$	$\lvert v(t) \rvert$	$a(t)$	Description of the particle's motion
0	2	-4	4	2	moving left, accelerating to the right
1	-1	-2	2	2	moving left but slower, still accelerating to the right
2	-2	0	0	2	stopped, still accelerating to the right
3	-1	2	2	2	moving to the right, accelerating to the right
4	2	4	4	2	moving to the right faster, still accelerating to the right
5	7	6	6	2	moving to the right faster yet, still accelerating to the right



It is too much work to do such work for complicated functions. We are generally interested when the particle is stopped or when it has no acceleration. We are also interested when the object is speeding up or slowing down. Realizing that an object's velocity is either, positive (moving right), negative (moving left) or zero (stopped) and an object's acceleration is either positive , negative, or zero (constant speed), we can now use a chart to determine all the possibilities of an object's motion as if you were looking at it from above.

	$a(t) > 0$	$a(t) < 0$	$a(t) = 0$
$v(t) > 0$	speeding up	slowing down	constant velocity right
$v(t) < 0$	slowing down	speeding up	constant velocity left
$v(t) = 0$	stopped, accelerating right	stopped, accelerating left	stopped, no acceleration

Example 3) A particle is moving along a horizontal line with position function $s(t) = t^2 - 6t + 5$. Do an analysis of the particle's direction (right, left), acceleration, motion (speeding up, slowing down), & position.

Step 1: $\boxed{v(t) = 2t - 6}$ So $v(t) = 0$ at $\boxed{t = 3}$

Step 2: Make a number line of $v(t)$ showing when the object is stopped and the sign and direction of the object at times to the left and right of that. Assume $t > 0$.

<--------------- -------------------->
$v(t)$ |----------------------0+++++++++++
0 3

Step 3: $\boxed{a(t) = 2}$. Does $a(t) = 0$? $\boxed{\text{No}}$

Step 4: Make a number line of $a(t)$ showing when the object has a positive and negative acceleration. Scale it exactly like the $v(t)$ number line.

$a(t)$ |+++++++++++++++++++++++
0

Step 5: Make a motion line directly below the last two putting all critical values, multiplying the signs and interpreting according to the chart above.

slowing down speeding up
motion |----------------------+++++++++++
0 3

----------------------------------->
<------------------

Step 6: Make a position graph to show where the object is at critical times and how it moves.

position ______t=3_ ___________t=0___________
-4 5

Example 4) A particle is moving along a horizontal line with position function $s(t)=t^3-9t^2+24t+4$. Do an analysis of the particle's direction, acceleration, motion (speeding up or slowing down), and position.

$v(t)=3t^2-18t+24=0$

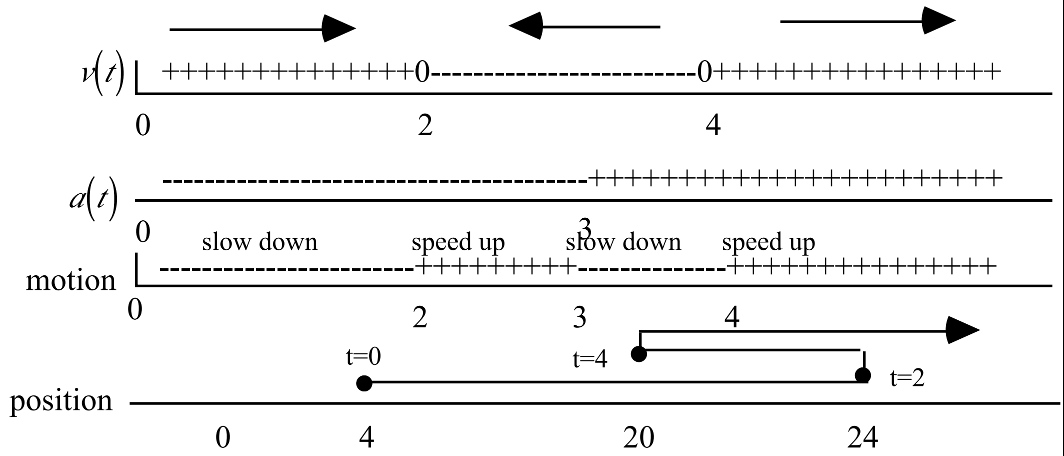
$v(t)=3(t^2-6t+8)=0$

$v(t)=3(t-2)(t-4)=0$

$t=2, t=4$

$a(t)=6t-18=0$

$t=3$



Note that the position graph is not like the other three graphs. It simply shows the position the object has with respect to the origin and critical times of its movement found by setting $v(t)$ and $a(t)=0$.

When an object is subjected to gravity, its position function is given by $s(t)=-16t^2+v_0t+s_0$, where t is measured in seconds, $s(t)$ is measured in feet, v_0 is the initial velocity (velocity at t = 0) and s_0 is the initial position (position at t = 0). The formula is given by $s(t)=-4.9t^2+v_0t+s_0$ if $s(t)$ is measured in meters.

From our original $s(t)=-16t^2+v_0t+s_0$, we can calculate the velocity function $\boxed{v(t)=-32t+v_o}$ and the acceleration function $\boxed{a(t)=-32}$. This is the acceleration due to gravity on earth.

When an object is thrown upward, it is subjected to gravity, We are usually interested how high the particle reaches and how fast it is going when it impacts the ground or water. Let us analyze what these mean:

When an object reaches its maximum height, what is its velocity? $\boxed{v=0}$

When an object hits the ground, what is its final position? $\boxed{s=0}$

So to find the maximum height of an object, set $v(t)=0$, solve for t, and find $s(t)$

So, to find the velocity of an object when it hits the ground, set $s(t)=0$, solve for t, and find $v(t)$

Example 5) A projectile is launched vertically upward from ground level with an initial velocity of 112 ft/sec.

a. Find the velocity and speed at t = 3 and t = 5 seconds.

b. How high will the projectile rise?

c. Find the speed of the projectile when it hits the ground.

a.

$s(t)=-16t^2+112t$

$v(t)=-32t+112$

$v(3)=16$ ft/sec speed $=16$ ft/sec

$v(5)=-48$ ft/sec speed $=48$ ft/sec

b.

$v(t)=-32t+112=0$

$32t=112$

$t=3.5$

$s(3.5)=-16(3.5)^2+112(3.5)$

$s=196$ ft

c.

$s(t)=-16t^2+112t=0$

$-16(t-7)=0$

$t=7$

$v(7)=-32(7)+112$

$v(7)=112$ ft/sec

Example 5) The equations for free fall at the surfaces of Mars, Earth, and Jupiter (*s* in meters, *t* in seconds) are: Mars: $s(t) = 1.86t^2$, Earth: $s(t) = 4.9t^2$, Jupiter: $s(t) = 11.44t^2$. How long would it take a rock, initially at rest in a space capsule over the planet, to reach a velocity of 16.6 m/sec?

Mars	Earth	Jupiter
$v(t) = 3.72t$	$v(t) = 9.8t$	$v(t) = 22.88t$
$3.72t = 16.6$	$9.8t = 16.6$	$22.88t = 16.6$
$t = 4.462$ sec	$t = 1.694$ sec	$t = .726$ sec

Example 6) A rock thrown vertically upward from the surface of the moon at a velocity of 24m/sec reaches a height of $s = 24t - 0.8t^2$ meters in *t* seconds.

a) Find the rock's velocity and acceleration as a function of time. (The acceleration in this case is the acceleration on the moon)

$$\boxed{v(t) = 24 - 1.6t \quad a(t) = -1.6}$$

b) How long did it take the rock to reach its highest point?

$$\boxed{v(t) = 24 - 1.6t = 0 \Rightarrow t = 15 \text{ sec}}$$

c) How high did the rock go?

$$s(15) = 24(15) - .8(15)^2$$
$$s(15) = 180 \text{ m}$$

d) How long did it take the rock to reach half its maximum height?

$$24t - 0.8t^2 = 90$$
$$.8t^2 - 24t + 90 = 0 \Rightarrow t = 4.393 \text{ sec}$$

e) How long was the rock aloft?

$$s(t) = 24t - 0.8t^2 = 0$$
$$8t(3 - .1t) = 0$$
$$t = 0, t = 30 \text{ sec}$$

e) Find the rock's speed when hitting the moon.

$$v(30) = 24 - 1.6(30)$$
$$v(t) = -24$$
$$\text{speed} = 24 \text{ m/sec}$$

Example 7) A ball is dropped from the top of the Washington Monument which is 555 feet high.

a) How long will it take for the ball to hit the ground?

$$s(t) = -16t^2 + 555 = 0$$
$$16t^2 = 555 \Rightarrow t = 5.89 \text{ sec}$$

b) Find the ball's speed at impact.

$$\boxed{|v(5.89)| = |-32(5.89)| = 188.48 \text{ ft/sec} \approx 128.5\text{mph}}$$

Example 8) Paul has bought a ticket on a special roller coaster at an amusement park which moves in a straight line. The position $s(t)$ of the car in feet after *t* seconds is given by: $s(t) = -.01t^3 + 1.2t^2, \quad 0 \le t \le 120$

a) Find the velocity and acceleration of the roller coaster after *t* seconds?

$$\boxed{v(t) = -.03t^2 + 2.4t \qquad a(t) = -.06t + 2.4}$$

b) When is the roller coaster stopped?

$$\boxed{-.03t^2 + 2.4t = 0 \rightarrow t = 0, t = 80 \text{ sec}}$$

c) When is Paul speeding up and slowing down?

$$-.06t + 2.4 = 0 \Rightarrow t = 40$$
$$\text{Speed up } (0,40),(80,120) \quad \text{Slow down } (40,80)$$

d) Where is Paul at critical times of his ride?

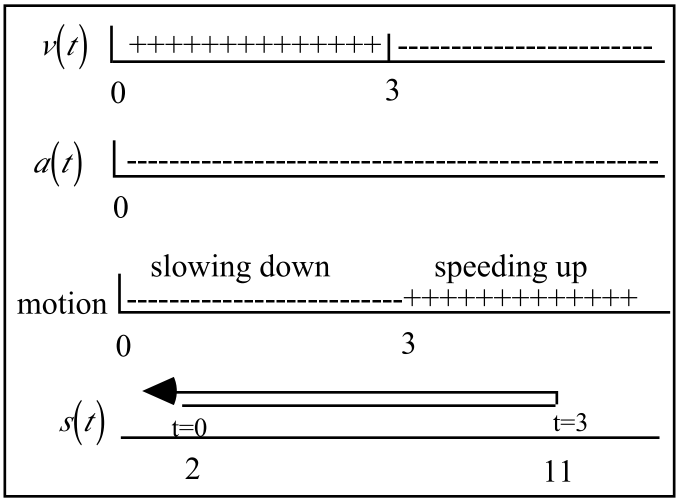
$$\boxed{t = 0, s = 0 \quad t = 80, s = 2560 \quad t = 120, s = 0}$$

Straight Line Motion - Homework

A particle is moving along a horizontal line with position function as given. Do an analysis of the particle's direction, acceleration, motion (speeding up or slowing down), and position.

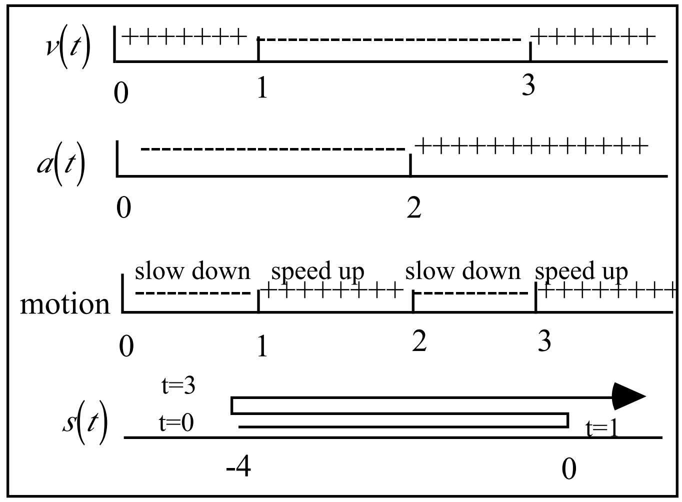
1. $s(t) = 2 + 6t - t^2$

$\boxed{v(t) = 6 - 2t = 0 \Rightarrow t = 3 \qquad a(t) = -2 \neq 0}$



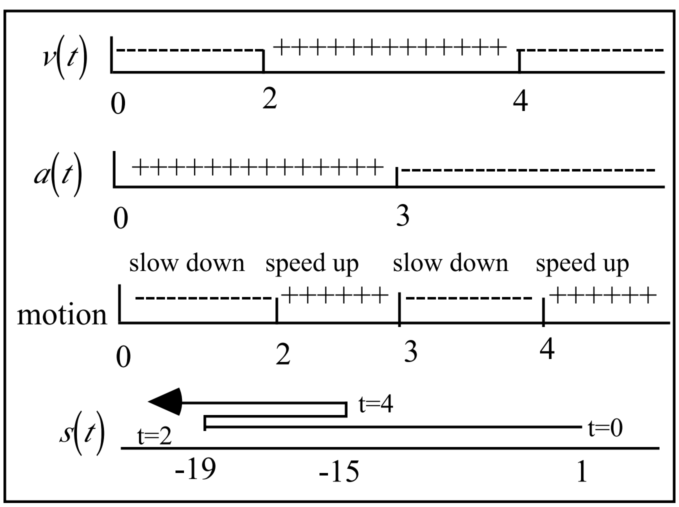
2. $s(t) = t^3 - 6t^2 + 9t - 4$

$\boxed{v(t) = 3t^2 - 12t + 9 \Rightarrow t = 1,3 \qquad a(t) = 6t - 12 \Rightarrow t = 2}$



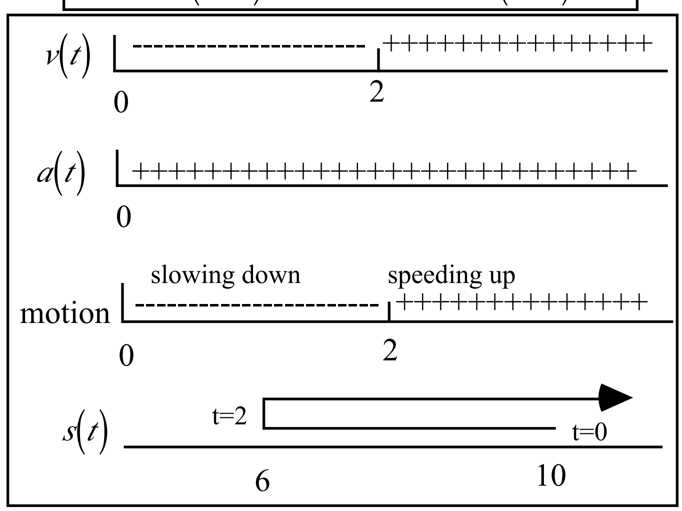
3. $s(t) = -t^3 + 9t^2 - 24t + 1$

$\boxed{v(t) = -3t^2 + 18t - 24 \Rightarrow t = 2,4 \qquad a(t) = -6t + 18 \Rightarrow t = 3}$



4. $s(t) = t + \dfrac{9}{t+1} + 1$

$\boxed{v(t) = 1 - \dfrac{9}{(t+1)^2} \Rightarrow t = 2 \qquad a(t) = \dfrac{-18}{(t+1)^3} \neq 0}$



5. A 45-caliber bullet fired straight up from the surface of the moon would reach a height of $s = 832t - 2.6t^2$ feet after t seconds. On Earth, in the absence of air, its height would be $s = 832t - 16t^2$ feet after t seconds. How long would it take the bullet to hit the ground in either case?

Earth

$$\boxed{\begin{array}{l} s(t) = t(832 - 16t) = 0 \\ 16t = 832 \Rightarrow t = 52 \text{ sec} \end{array}}$$

Moon

$$\boxed{\begin{array}{l} s(t) = t(832 - 2.6t) = 0 \\ 2.6t = 832 \Rightarrow t = 320 \text{ sec} \end{array}}$$

6. A ball fired downward from a height of 112 feet hits the ground in 2 seconds. Find its initial velocity.

$$\boxed{\begin{array}{l} s(t) = -16t^2 + v_o t + 112 = 0 \\ -64 + 2v_o + 112 = 0 \Rightarrow 2v_o = -48 \\ v_o = -24 \text{ ft/sec} \end{array}}$$

7. A projectile is fired vertically upward (earth) from ground level with an initial velocity of 16 ft/sec.

a. How long will it take for the projectile to hit the ground?

$$\boxed{\begin{array}{l} s(t) = -16t^2 + 16t = 0 \\ -16t(t-1) = 0 \\ t = 1 \text{ sec} \end{array}}$$

b. How high will the projectile get?

$$\boxed{\begin{array}{l} v(t) = -32t + 16 = 0 \\ 32t = 16 \Rightarrow t = .5 \\ s(.5) = -16(.5)^2 + 16(.5) = 4 \text{ ft} \end{array}}$$

8. A helicopter pilot drops a package when the helicopter is 200 ft. above the ground, rising at 20 ft/sec.

a. How long will it take for the package to hit the ground?

$$\boxed{\begin{array}{l} s(t) = -16t^2 + 20t + 200 = 0 \\ s(t) = -4\left(4t^2 - 5t - 50\right) \Rightarrow t = 4.215 \text{ sec} \end{array}}$$

b. What is the speed of the package at impact?

$$\boxed{\begin{array}{l} v(t) = -32t + 20 \\ \left|v(4.215)\right| = \left|-32(4.215) + 20\right| = 114.891 \text{ ft/sec} \end{array}}$$

9. A man drops a quarter from a bridge. How high is the bridge if the quarter hits the water 4 seconds later?

$$\boxed{\begin{array}{l} s(t) = -16t^2 + s_0 = 0 \\ s(t) = -16(4)^2 + s_0 = 0 \\ s_o = 256 \text{ ft} \end{array}}$$

10. A projectile fired upward from ground level is to reach a maximum height of 1,600 feet. What is its initial velocity?

$$\boxed{\begin{array}{l} s(t) = -16t^2 + v_0 t = 1600 \\ v(t) = -32t + v_0 = 0 \\ v_0 = 32t \\ -16t^2 + (32t)t = 1600 \\ 16t^2 = 1600 \\ t = 10 \text{ sec} \\ v_0 = 32(10) = 320 \text{ ft/sec} \end{array}}$$

11. A projectile is fired vertically upward with an initial velocity of 96 ft/sec from a tower 256 feet high.

a. How long will it take for the projectile to reach its maximum height.

$$\boxed{\begin{array}{l} s(t) = -16t^2 + 96t + 256 \\ v(t) = -32t + 96 = 0 \Rightarrow t = 3 \text{ sec} \end{array}}$$

b. What is its maximum height?

$$\boxed{\begin{array}{l} s(t) = -16t^2 + 96t + 256 \\ v(t) = -32t + 96 = 0 \Rightarrow t = 3 \text{ sec} \\ s(3) = 400 \text{ ft} \end{array}}$$

c. How long will it take the projectile to reach its starting height on the way down?

$$\boxed{\begin{array}{l} s(t) = -16t^2 + 96t + 256 = 256 \\ -16t(t-6) = 0 \Rightarrow t = 6 \text{ sec} \end{array}}$$

d. What is the velocity when it passes the starting point on the way down?

$\boxed{-96 \text{ ft/sec}}$

e. How long will it take to hit the ground?

$$\boxed{\begin{array}{l} s(t) = -16t^2 + 96t + 256 = 0 \\ -16(t^2 - 6t - 16) = 0 \Rightarrow t = 8 \text{ sec} \end{array}}$$

f. What will be its speed when it impacts the ground?

$\boxed{|v(8)| = |-32(8) + 96| = 160 \text{ ft/sec}}$

12. John's car runs out of gas as it goes up a hill. The car rolls to a stop then starts rolling backwards. As it rolls, its displacement $d(t)$ in feet from the bottom of the hill at t seconds since the car ran out of gas is given by: $d(t) = 125 + 31t - t^2$.

a. When is his velocity positive? What does this mean in real world terms?

$$\boxed{\begin{array}{l} v(t) = 31 - 2t > 0 \\ 0 \le t < 15.5 \text{ sec} \;\; - \text{ going up the hill} \end{array}}$$

b. When did the car start to roll backwards? How far was it from the bottom of the hill at that time?

$$\boxed{\begin{array}{l} v(t) = 31 - 2t < 0 \\ 15.5 \text{ sec} \;\; - \; d(15.5) = 365.25 \text{ ft} \end{array}}$$

c. If John keeps his foot off the brake, when will he be at the bottom of the hill?

$\boxed{125 + 31t - t^2 = 0 \Rightarrow t \approx 34.612 \text{ sec}}$

d. How far was John from the bottom of the hill when he ran out of gas?

$\boxed{d(0) = 125 \text{ ft}}$

13. Ray is a sky-diver. When he free-falls, his downward velocity $v(t)$ feet per second is a function of t seconds from the time of the jump is given by: $v(t) = 251(1 - 0.88^t)$ measured in ft/sec.. Plot $v(t)$ and $a(t)$ on your calculator for the first 30 seconds of his dive.

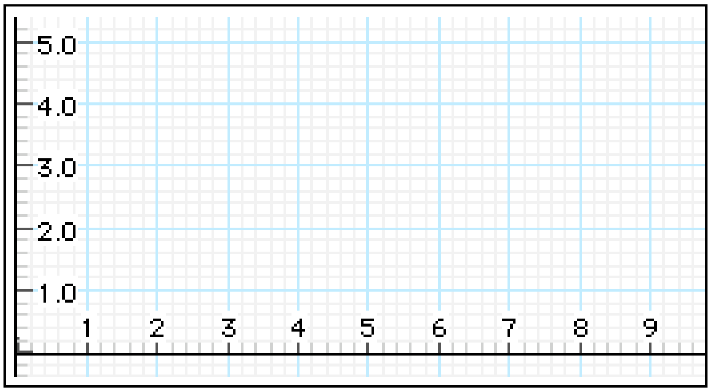
a. What is Ray's acceleration when he first jumps? Why does the acceleration decrease over time?

$\boxed{32.086 \text{ ft/sec}^2 \;\; - \text{ air resistance}}$

b. What appears to be the terminal velocity, $\lim\limits_{t \to \infty} v(t)$?

$\boxed{251 \text{ ft/sec}}$

Rolle's and the Mean Value Theorem - Classwork



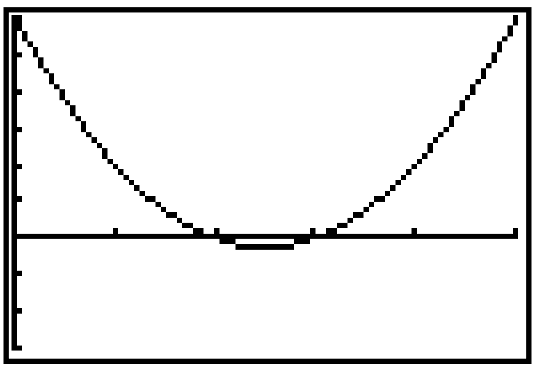
On the graph to the left, plot a point at (2, 2) and another point at (7, 2). Now draw 3 graphs of a differentiable function that starts at (2, 2) and ends at (7, 2). Is it possible that there is at least one point on your graphs for which the derivative is not zero? Is it possible that you can draw a graph for which there is not one point where the derivative is not zero?

Rolle's Theorem

Let f be continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b). If $f(a)=f(b)$, then there is at least one number c in (a,b) such that $f'(c)=0$.

What Rolle's theorem is that on a differentiable curve between two values of the same height, there must be at least one point in between where the tangent line is horizontal. In layman's terms, "what goes up must come down and what goes down must come back up."

Example 1) Show that Rolle's theorem holds between the intercepts of $f(x)=x^2-5x+6$.



What are the x-intercepts of $f(x)=x^2-5x+6$? $\boxed{x=2,3}$

Is the curve continuous? $\boxed{\text{yes}}$ Is the curve differentiable? $\boxed{\text{yes}}$ Why? $\boxed{\text{polynomials are differentiable}}$

So you may now use Rolle's theorem. Do so.

$$\boxed{2x-5=0 \Rightarrow x=\frac{5}{2}}$$

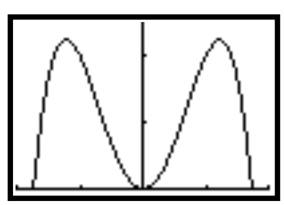
Example 2) Find all values between the x-intercepts for which Rolle's theorem holds for the roots of $f(x)=3x^2-x^4$. Confirm using your calculator.

$3x^2-x^4=0 \Rightarrow x=0,\pm\sqrt{3}$ so $a=-\sqrt{3}, b=\sqrt{3}$

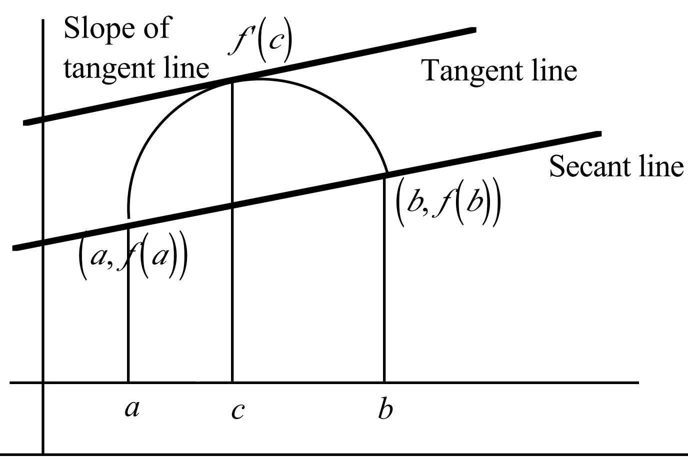
$6x-4x^3=0$

$x(6-4x^2)=0$

$x=0, x=\pm\sqrt{\frac{3}{2}}$ (Answer does not include $x=0$ as c does not include endpoints)



You are driving in a car traveling at 50 mph and you pass a police car. Four minutes later, you pass a second police car and you are traveling at 50 mph. The distance between the two police cars is five miles. The second police car nails you for speeding. How can he prove that you were speeding? **If avg vel = 75 mph, you must travel 75 mph at least once in that time interval.**

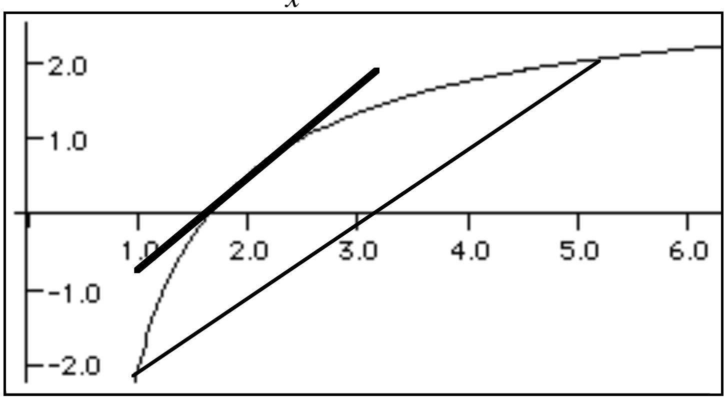


The Mean - Value Theorem

If f is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), then there exists a number c on (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Example 3) Given $f(x) = 3 - \dfrac{5}{x}$. Find the value of c in the interval (1, 5) that satisfies the mean value theorem.



To the left, is the graph of the function.

Visually, what is the mean value theorem trying to find? Draw it in.

Let's use the mean value theorem to find the solution.

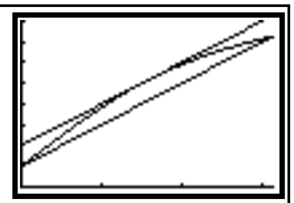
$$\boxed{\frac{f(5) - f(1)}{5 - 1} = \frac{-5}{x^2} \Rightarrow \frac{-2 - 2}{4} = \frac{-5}{x^2} = -1 \Rightarrow x^2 = 5 \Rightarrow x = \pm\sqrt{5} \text{ ... in interval } x = \sqrt{5}}$$

Example 4) Find the equation of the tangent line to the graph of $f(x) = 2x + \sin x + 1$ on $(0, \pi)$ at the point which is the solution of the mean-value theorem. Confirm by calculator.

$$\frac{f(\pi) - f(0)}{\pi - 1} = 2 + \cos x \Rightarrow \frac{2\pi + 1 - 1}{\pi} = 2 = 2 + \cos x$$

$$\cos x = 0 \Rightarrow x = \frac{\pi}{2} \Rightarrow y = \pi + 1 + 1 = \pi + 2$$

$$y - (\pi + 2) = 2\left(x - \frac{\pi}{2}\right) \Rightarrow y = 2x + 2$$



Example 5) Why can't you use the mean-value theorem for $f(x) = x^{2/3}$ on (-1, 1)?

$$\boxed{f(x) \text{ not differentiable at } x = 0}$$

Rolle's and the Mean Value Theorem - Homework

For the exercises below, determine whether Rolle's theorem can be applied to the function in the indicated interval. If Rolle's Theorem can be applied, find all values of x that satisfy Rolle's Theorem.

1. $f(x) = x^2 - 4x$ on $[0,4]$

$$\boxed{\begin{array}{l} 2x - 4 = 0 \\ x = 2 \end{array}}$$

2. $f(x) = x^2 - 11x + 30$ on $[5,6]$

$$\boxed{\begin{array}{l} 2x - 11 = 0 \\ x = \dfrac{11}{2} \end{array}}$$

3. $f(x) = (x-2)(x-3)(x-4)$ on $[2,4]$

$$\boxed{\begin{array}{l} x^3 - 9x^2 + 26x - 24 = 0 \\ 3x^2 - 18x + 26 = 0 \\ x = \dfrac{18 \pm \sqrt{12}}{6} = 3.577, 2.423 \end{array}}$$

4. $f(x) = (x+4)^2(x-3)$ on $[-4,3]$

$$\boxed{\begin{array}{l} x^3 + 5x^2 - 8x - 48 = 0 \\ 3x^2 + 10x - 8 = 0 \\ (x+4)(3x-2) = 0 \Rightarrow x = \dfrac{2}{3} \end{array}}$$

5. $f(x) = 4 - |x-2|$ on $[-3,7]$

$\boxed{\text{Not differentiable at } x = 2}$

6. $f(x) = \sin x$ on $[0,2\pi]$

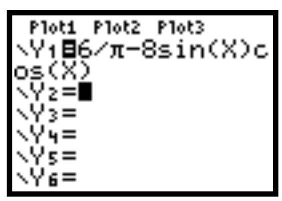
$$\boxed{\begin{array}{l} \cos x = 0 \\ x = \dfrac{\pi}{2}, \dfrac{3\pi}{2} \end{array}}$$

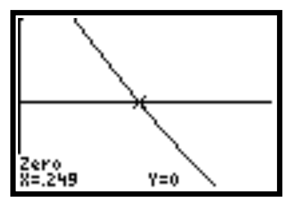
7. $f(x) = \cos 2x$ on $\left[\dfrac{\pi}{3}, \dfrac{2\pi}{3}\right]$

$$\boxed{\begin{array}{l} -2\sin x = 0 \\ 2x = 0 \Rightarrow 2x = 0, \pi, 2\pi .... \\ x = 0, \dfrac{\pi}{2}, \pi, ... \\ \text{In interval}: x = \dfrac{\pi}{2} \end{array}}$$

8. $f(x) = \dfrac{6x}{\pi} - 4\sin^2 x$ on $\left[0, \dfrac{\pi}{6}\right]$

$$\boxed{\dfrac{6}{\pi} - 8\sin x \cos x = 0}$$





For the exercises below, apply the mean value theorem to $f(x)$ on the indicated interval. Find all values of c which satisfy the mean value theorem.

9. $f(x) = x^2$ on $[-1,2]$

$$\boxed{\begin{array}{l} \dfrac{4-1}{3} = 2x \\ x = \dfrac{1}{2} \end{array}}$$

10. $f(x) = x^3 - x^2 - 2x$ on $[-1,1]$

$$\boxed{\begin{array}{l} \dfrac{-2-0}{2} = 3x^2 - 2x - 2 \\ 3x^2 - 2x - 1 = 0 \\ (3x+1)(x-1) = 0 \\ x = \dfrac{-1}{3} \end{array}}$$

11. $f(x) = \dfrac{x+2}{x}$ on $\left[\dfrac{1}{2}, 2\right]$

$$\boxed{\begin{array}{l} \dfrac{2-5}{1.5} = \dfrac{-2}{x^2} \\ -2x^2 = -2 \\ x = \pm 1, x = 1 \end{array}}$$

12. $f(x) = \sqrt{x-3}$ on $[3,7]$

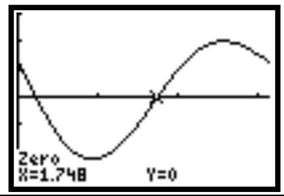
$$\boxed{\begin{array}{l} \dfrac{2-0}{4} = \dfrac{1}{2\sqrt{x-3}} \\ 4\sqrt{x-3} = 4 \\ (x-3) = 1 \\ x = 4 \end{array}}$$

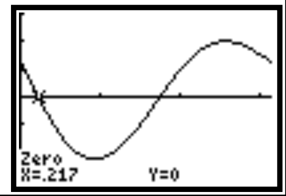
13. $f(x) = x^3$ on $[0,1]$

$$\boxed{\begin{array}{l} \dfrac{1-0}{1} = 3x^2 \\ x^2 = \dfrac{1}{3} \\ x = \pm\sqrt{\dfrac{1}{3}} \Rightarrow x = \sqrt{\dfrac{1}{3}} \end{array}}$$

14. $f(x) = 2\cos x + \cos 2x$ on $[0, \pi]$

$$\dfrac{-1-3}{\pi} = -2\sin x - 2\sin 2x$$





15. A trucker handed in a ticket at a toll booth showing that in 2 hours the truck had covered 159 miles on a toll road in which the speed limit was 65 mph. The trucker was cited for speeding. Why?

Average velocity is 79.5 mph. There must be some time when the trucker was traveling at 75 mph.

16. A marathoner ran the 26.2 mile New York marathon in 2 hours, 12 minutes. Show that at least twice, the marathoner was running at exactly 11 mph.

Avg vel $= \dfrac{26.2}{2.2} = 11.909$ mph. Runner had to reach 11 mph at least twice, once when speeding up to 11.909, once when slowing down

17. The order and transportation cost C of bottles of Pepsi is approximated by the function:

$C(x) = 10{,}000\left(\dfrac{1}{x} + \dfrac{x}{x+3}\right)$ where x is the order size of bottles of Pepsi in hundreds.

According to Rolles's Theorem, the rate of change of cost must be zero for some order size in the interval [3,6]. Find that order size.

$$\dfrac{0-0}{6-3} = 10000\left(\dfrac{-1}{x^2} + \dfrac{3}{(x+3)^2}\right) \Rightarrow \dfrac{1}{x^2} = \dfrac{3}{(x+3)^2}$$

$3x^2 = x^2 + 6x + 9 \Rightarrow 2x^2 - 6x - 9 = 0 \Rightarrow x = 4.098\ldots$ order size $= 410$

18. A car company introduces a new car for which the number of cars sold S is the function:

$S(t) = 300\left(5 - \dfrac{9}{t+2}\right)$ where t is the time in months.

a) Find the average rate of cars sold over the first 12 months. $\boxed{\text{96.421 cars}}$

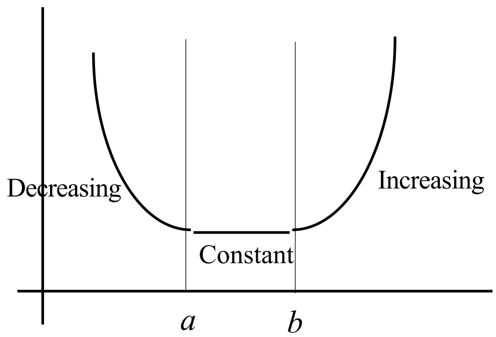
b) During what month does the average rate of cars sold equal the rate of change of cars sales?

$$\boxed{\dfrac{2700}{(t+2)^2} = 96.421 \Rightarrow t = 3.291 - \text{April}}$$

Function Analysis - Classwork

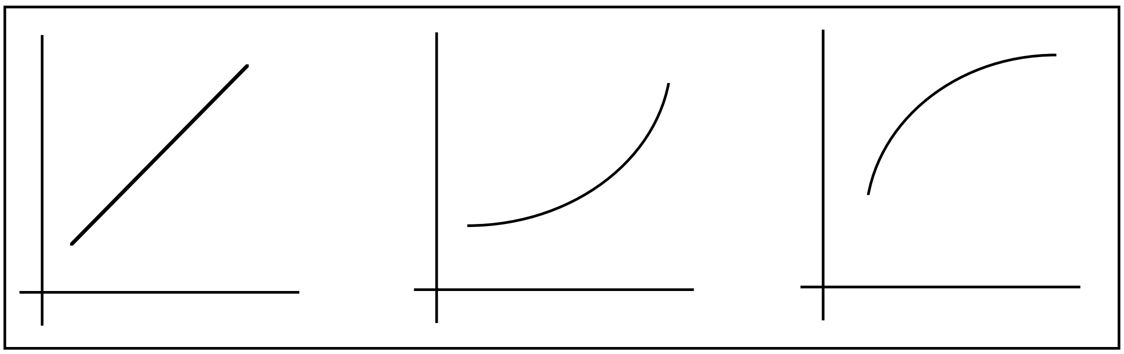
We now turn to analyzing functions via calculus. We did so in precalculus by determining the zeros of the function (where it crosses the *x*-axis) and the sign of the function between zeros. That gave us some information about the function but little clue as to its actual shape. Calculus will provide the missing pieces to the puzzle.

We will define a function as increasing if, as *x* moves to the right, the *y*-value goes up. A function is decreasing if, as *x* moves to the right, the *y*-value goes down. A function is constant if, as *x* moves to the right, the *y*-value doesn't change.



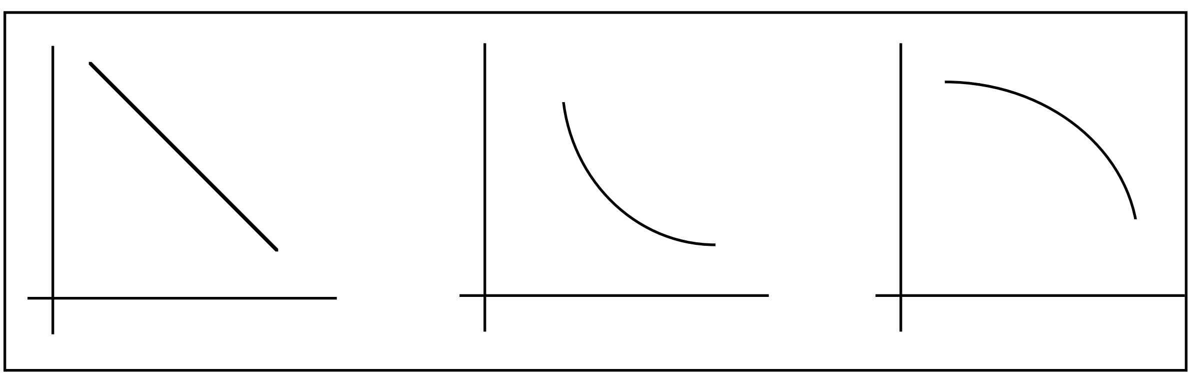
$f(x)$ is decreasing on $(-\infty, a)$
$f(x)$ is constant on $[a,b]$
$f(x)$ is increasing on (b,∞)

Increasing curves come in 3 varieties as shown. In each case, let's draw tangent lines at points along the curve.



What is true about the slope of the tangent lines (i.e. the derivative) at all of these points? all positive

Decreasing curves come in 3 varieties as shown. In each case, let's draw tangent lines at points along the curve.

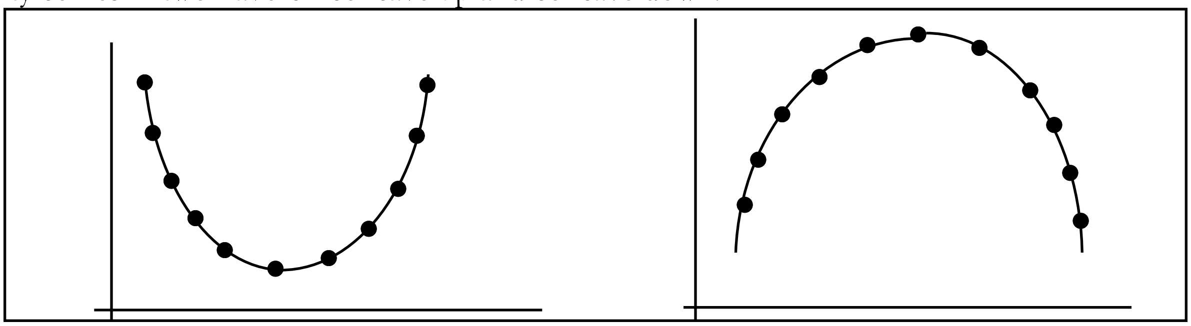


What is true about the slope of the tangent lines (i.e. the derivative) at all of these points? all negative

So we can make the following statetments about increasing and decreasing functions: Let f be a function that is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b),

If $f'(x) > 0$ for all x in (a,b), then f is increasing on $[a,b]$.
If $f'(x) < 0$ for all x in (a,b), then f is decreasing on $[a,b]$.
If $f'(x) = 0$ for all x in (a,b), then f is constant on $[a,b]$.

When we examine functions that have curves to them, we will define their curvature in terms of concavity. Concavity comes in two flavors - concave up and concave down.



We call this curve **concave up**. The term I use is "holds water." If water is is poured into this structure, it will collect and "hold."

On the concave up curve, draw tangent lines at the points shown. What is true about the slopes of these lines as you go left to right. $\boxed{\text{slopes increasing}}$

We call this curve **concave down**. The term I use is "spills water." If water is poured into this structure, it will spill off.

On the concave down curve, draw tangent lines at the points shown. What is true about the slopes of these lines as you go left to right. $\boxed{\text{slopes decreasing}}$

So we can define concavity as follows: If f is differentiable on an interval I. The graph of f is concave up on I if f' is increasing on the interval and concave down on I if f' is decreasing on the interval.

When you think "slope of tangent line", you think $\boxed{\text{derivative}}$

When you think increasing, you think $\boxed{\text{derivative}}$

When you think decreasing, you think $\boxed{\text{derivative}}$

So we are talking derivatives of derivatives, which means $\boxed{\text{2nd derivative}}$ So concavity is related to the sign of the second derivative, $f''(x)$. If $f''(x) > 0$, for all x in an interval, then the graph of f is concave up on the interval. If $f''(x) < 0$ for all x in an interval, then the graph of f is concave down on the interval. Straight lines have no concavity (concavity = 0). That makes sense as a line has no curvature. All of this can be summarized: When you are given information about $f'(x)$ and $f''(x)$, you can determine the shape of $f(x)$.

	$f'(x) > 0$	$f'(x) < 0$	$f'(x) = 0$
$f''(x) > 0$	$f(x)$ increasing, concave up	$f(x)$ decreasing, concave up	relative minimum, concave up
$f''(x) < 0$	$f(x)$ increasing, concave down	$f(x)$ decreasing, concave down	relative maximum, concave down
$f''(x) = 0$	$f(x)$ increasing, inflection point	$f(x)$ decreasing, inflection point	$f(x)$ levels off possible inflection point

Here are some terms which you must know. They are basic to any calculus course.

Critical Values (points): x-values on the function where the function has a slope equal to zero or where the function is not differentiable. Visually, it is where there is a horizontal tangent line or a vertical tangent line.

Stationary Point: A point on the function where there is a horizontal tangent at the x-value.

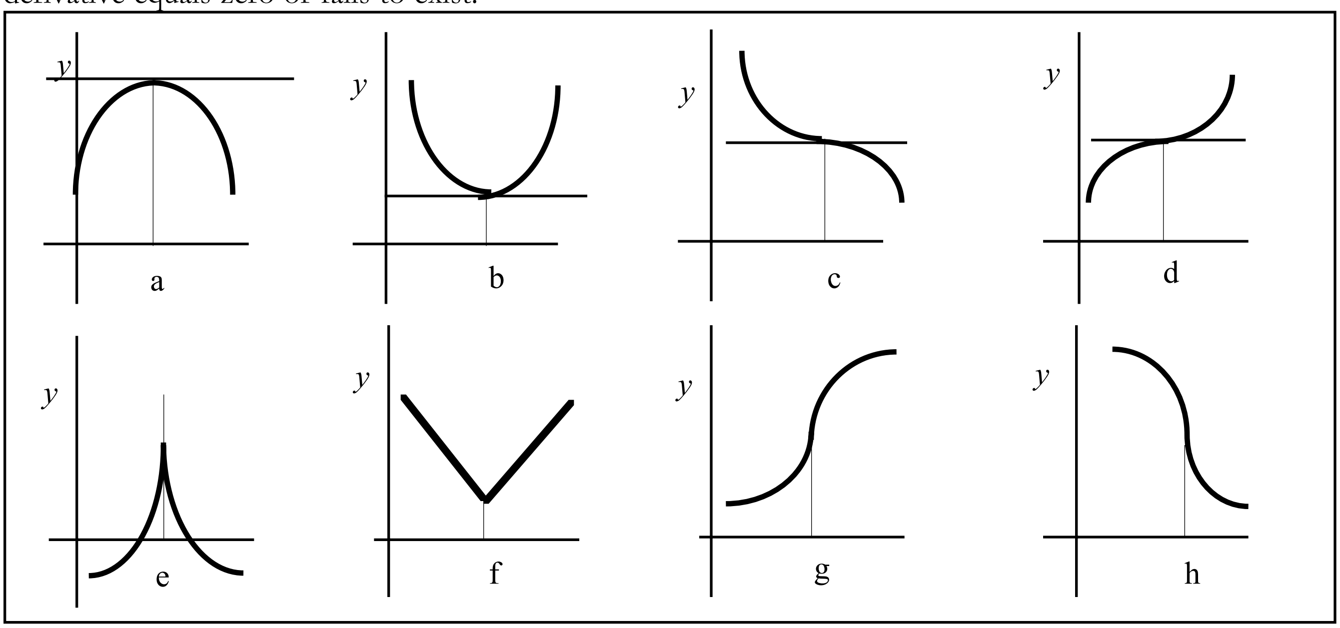
Relative Minimum: Informal definition: the bottom of a hill. Relative minimums occur where the curves switches from decreasing to increasing. The x-value is **where** the relative minimum occurs. The y-value is **what** the relative minimum is.

Relative Maximum: Informal definition: the top of a hill. Relative maximums occur where the curve switches from increasing to decreasing. The x-value is **where** the relative maximum occurs. The y-value is **what** the relative maximum is.

Relative Extrema: Either a relative minimum or relative maximum.

Absolute maximum or minimum. The highest (lowest) point on the curve. The x-value is **where** the absolute maximum(minimum) occurs. The y-value is **what** the absolute maximum (minimum) is. The absolute minimum or maximum must occur at relative extrema or at the endpoints of an interval.

Inflection point: The x-value where the curve switches concavity. Inflection points can occur where the second derivative equals zero or fails to exist.

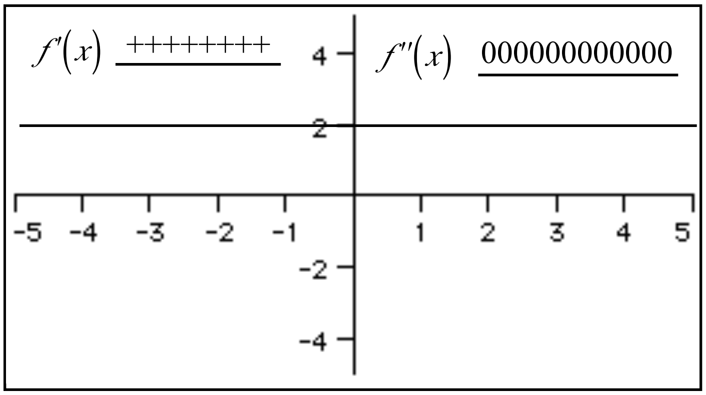


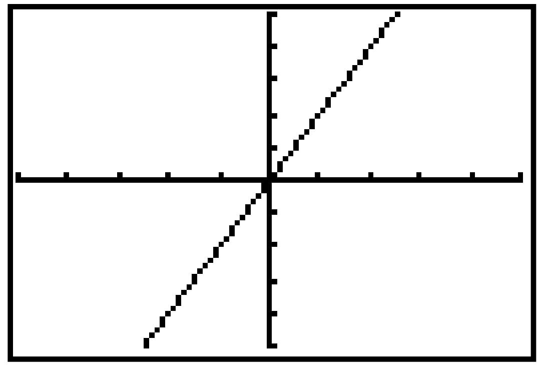
Example 1) For each term, determine if it is applicable at the x-values a - h.

	Critical Point	Relative Minimum	Relative Maximum	Stationary Point	Inflection Point
a	*		*	*	
b	*	*		*	
c	*			*	*
d	*			*	*
e	*		*		
f	*	*			
g	*				*
h	*				*

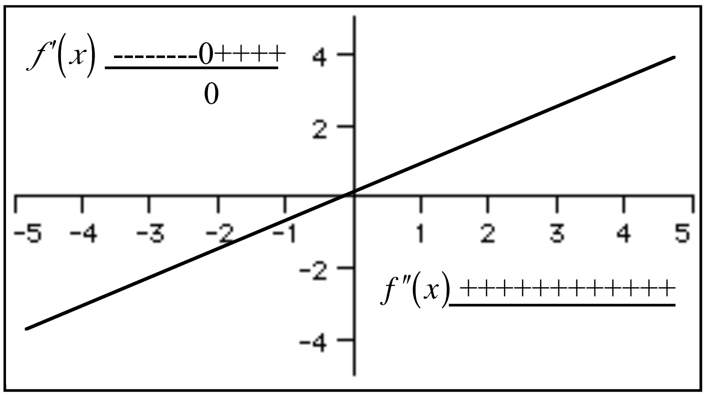
Example 2) You are given a graph of $f'(x)$. Draw a picture of a possible $f(x)$.

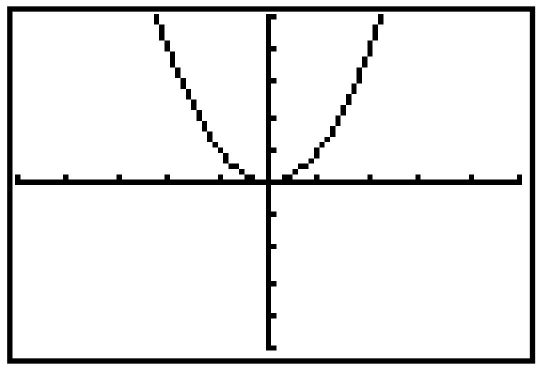
a. $f'(x)$ possible $f(x)$



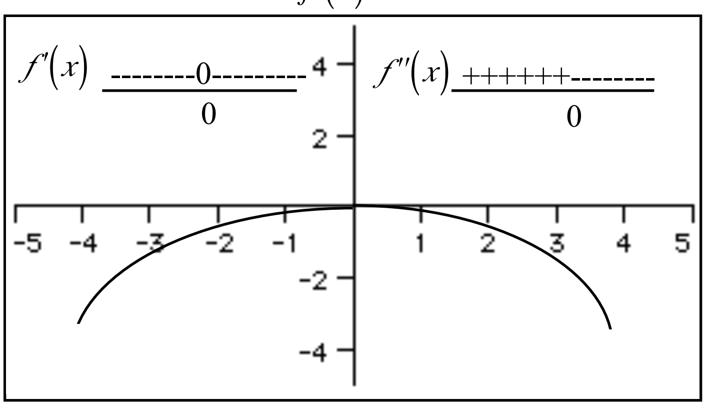


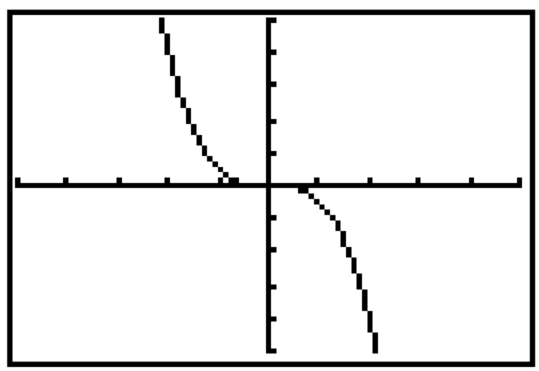
b. $f'(x)$ possible $f(x)$



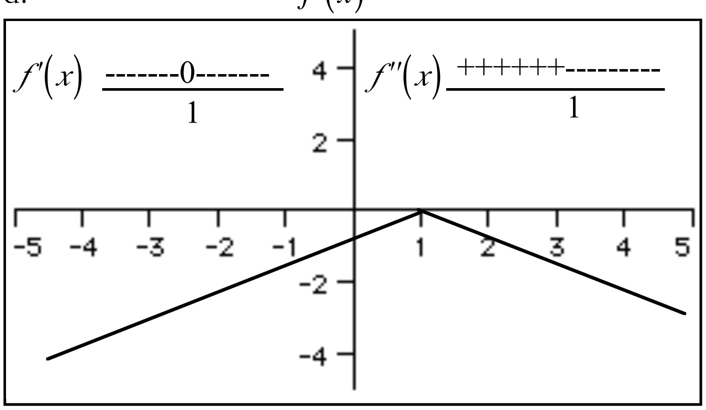


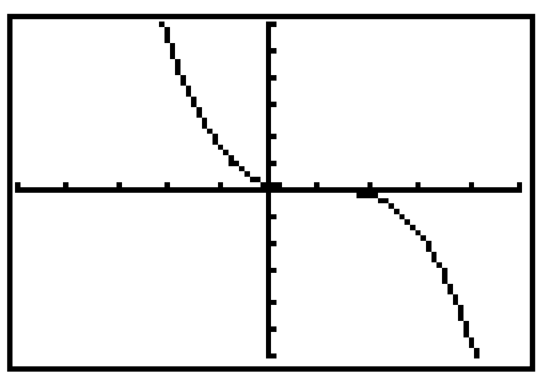
c. $f'(x)$ possible $f(x)$



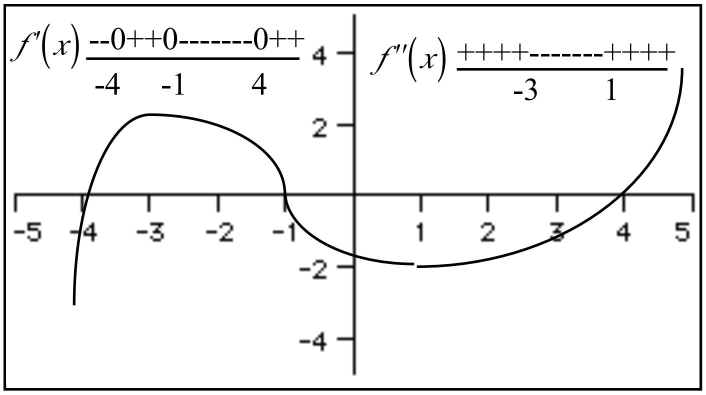


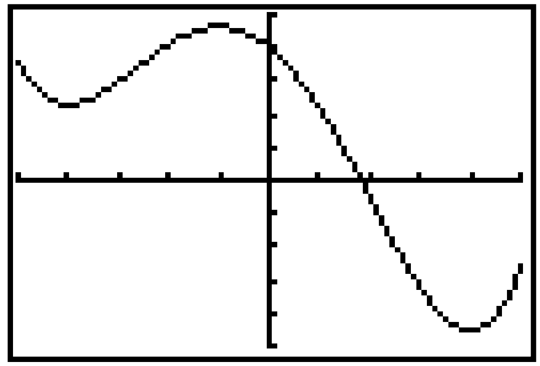
d. $f'(x)$ possible $f(x)$



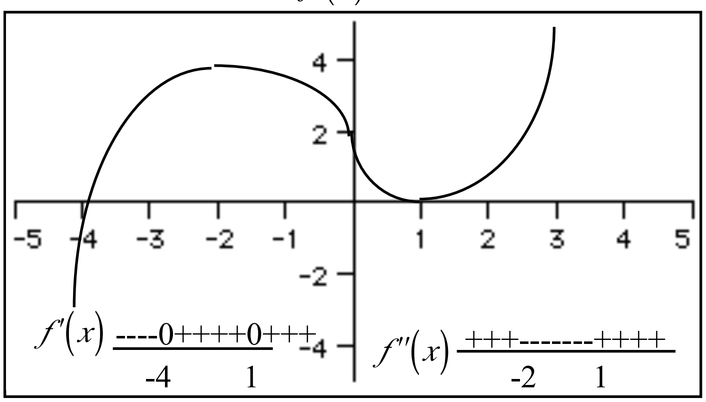


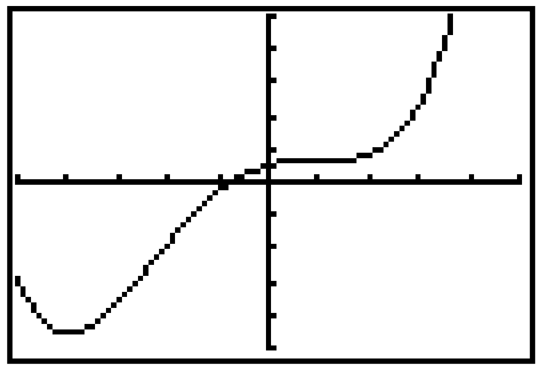
e. $f'(x)$ possible $f(x)$



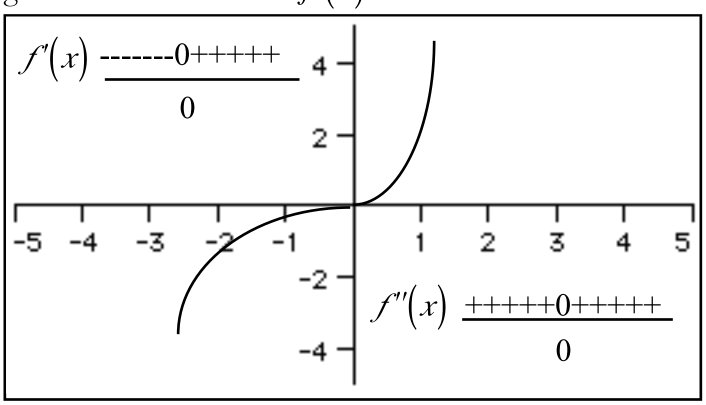


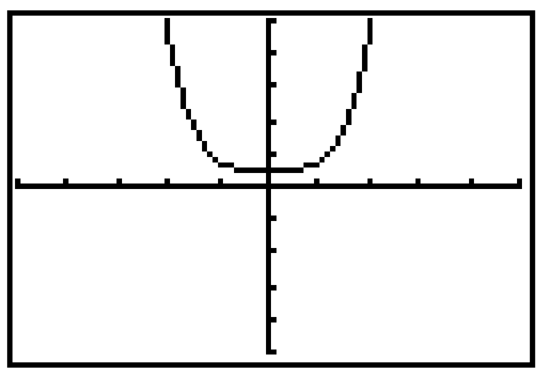
f. $f'(x)$ possible $f(x)$



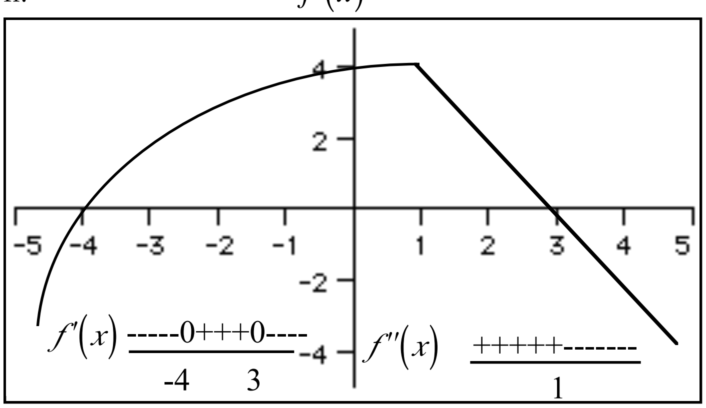


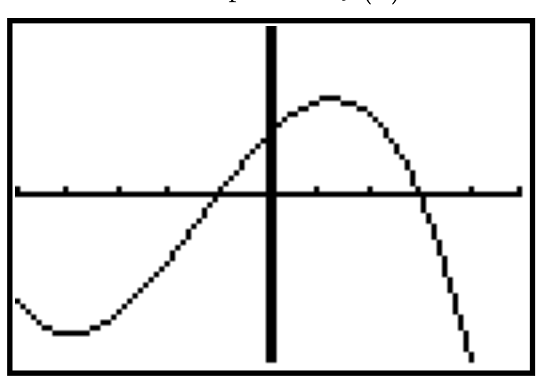
g. $f'(x)$ possible $f(x)$





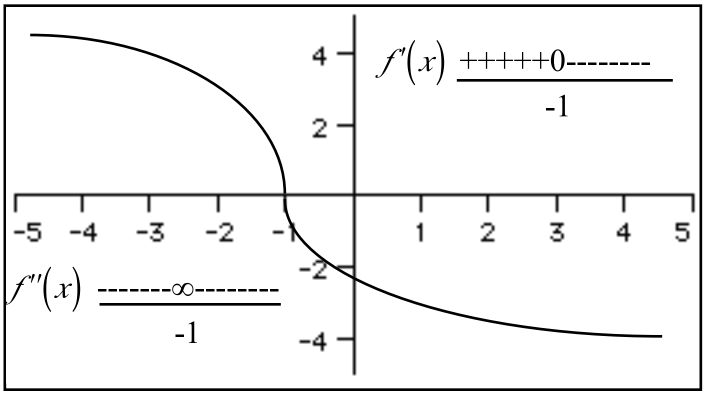
h. $f'(x)$ possible $f(x)$

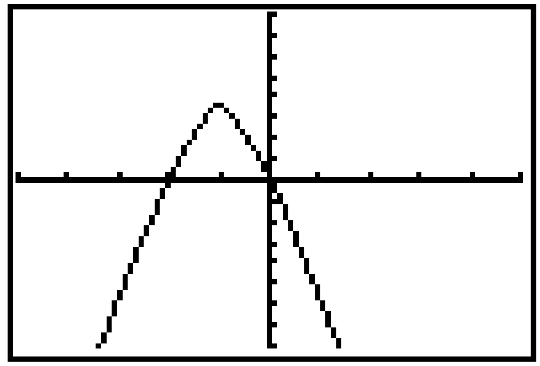




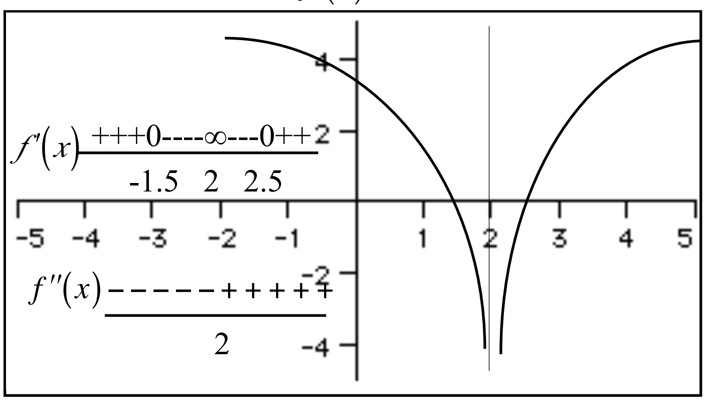
Example 3) You are given a graph of $f'(x)$. Draw a picture of a possible $f(x)$.

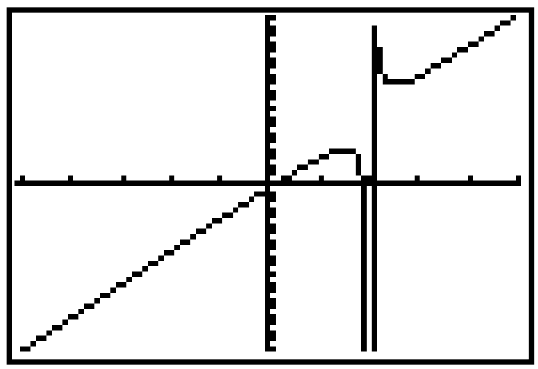
a. $f'(x)$ possible $f(x)$



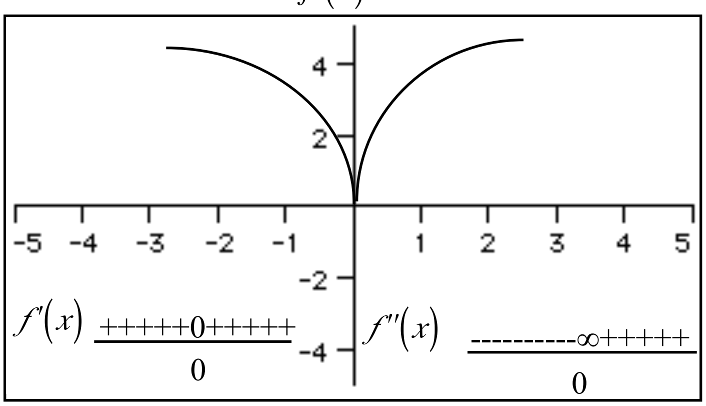


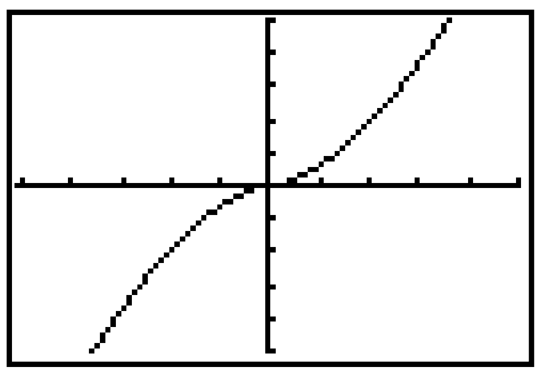
b. $f'(x)$ possible $f(x)$



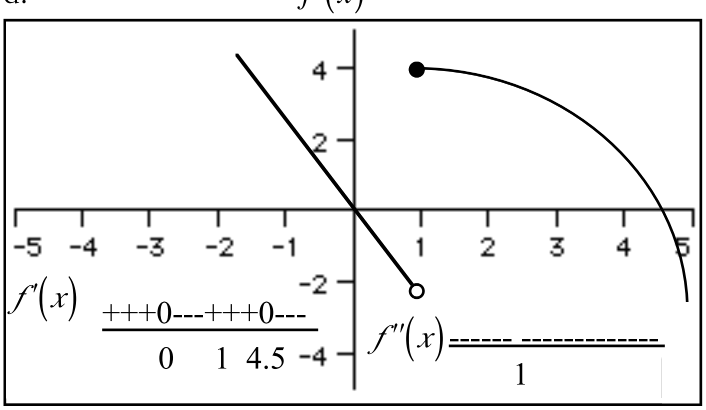


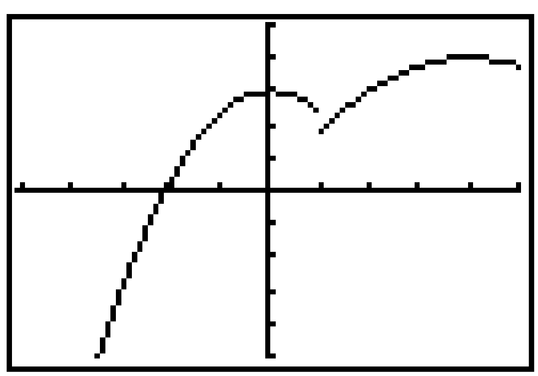
c. $f'(x)$ possible $f(x)$





d. $f'(x)$ possible $f(x)$

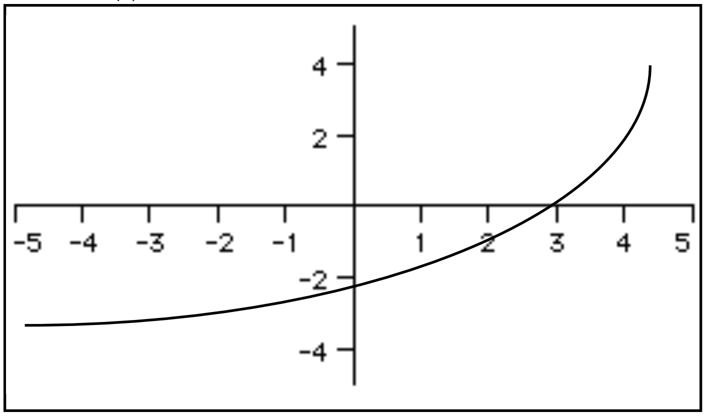


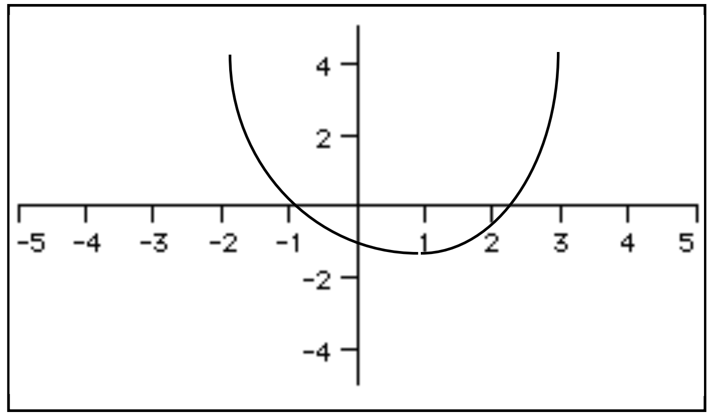


Example 4) Sketch a possible $f(x)$ given the following information.

a. $f'(x) > 0$
$f''(x) > 0$.
$f(1) = -2$

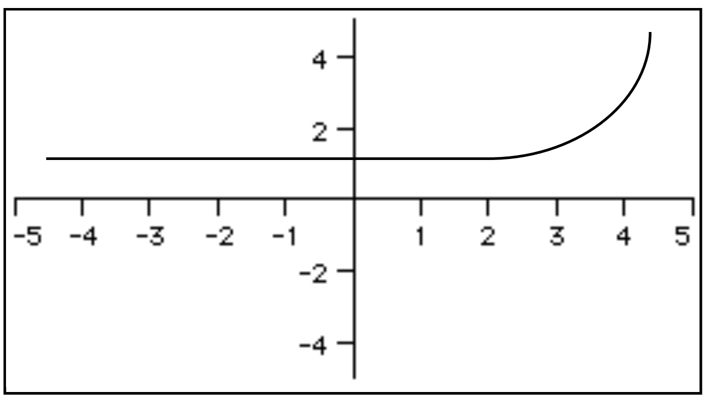
b. $f'(x) > 0, x > 1, \quad f'(x) < 0, x < 1, \quad f'(1) = 0$
$f''(x) > 0, \quad f(1) = -1$

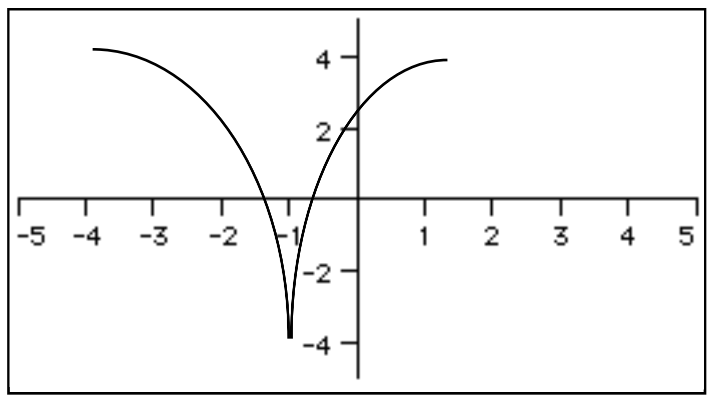




c. $f'(x) > 0, x > 2, \quad f'(x) = 0, x \le 2$
$f''(x) > 0, x > 2 \quad f(2) = 1$

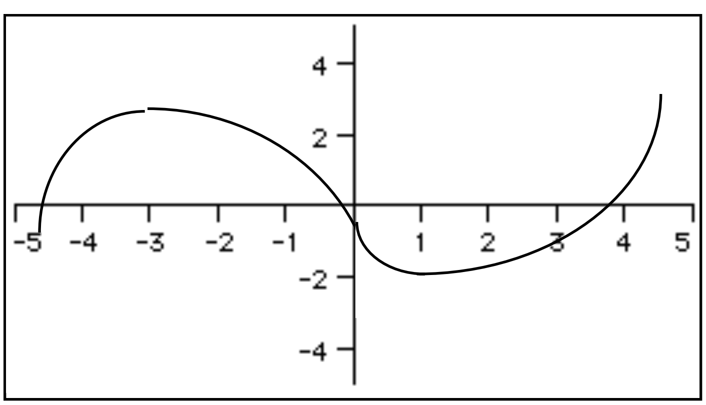
d. $f'(x) > 0, x > -1, \quad f'(x) < 0, x < -1$
$f''(x) < 0, \quad f(-1) = -4$

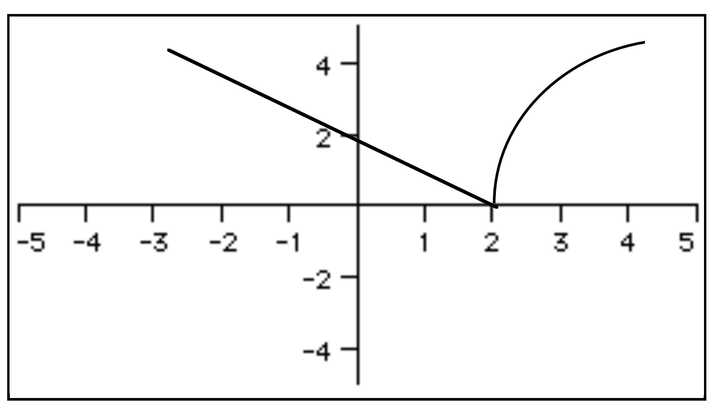




e. $f'(x) > 0, x > 1, f'(x) > 0, x < -3, f'(x) < 0, -3 < x < 1$
$f'(-3) = 0, \quad f'(1) = 0$
$f''(x) < 0, x < 0, \quad f''(x) > 0, x > 0$

f. $f'(x) > 0, x > 2, f'(x) = -1, x < 2, f'(2)$ DNE
$f''(x) < 0, x > 2, \quad f(2) = 0$

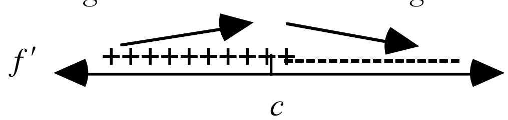




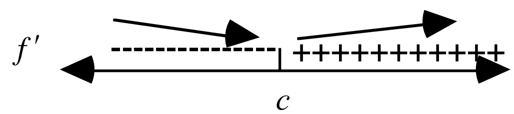
Now that we can determine the graph of a function by examining its first and second derivatives, we now attack the problem from algebraically. We wish to graph some function $f(x)$ by finding its relative maximum and minimum (extrema). With the advent of graphing calculators, a lot of this work is now done by technology and the act of sheer graphing of functions is being reduced on the A.P. exam.

To determine relative extrema of a function , do the following steps.

1. Find the derivative $f'(x)$. It is best that it be in a fraction form.
2. Find the critical values by setting $f'(x) = 0$ or by finding where $f'(x)$ is undefined. If $f'(x)$ is in fraction form, you merely set the numerator and denominator of $f'(x) = 0$ and solve.
3. Make a sign chart for $f'(x)$. Be sure you label it. On it, you will place every critical value *c* found above.
4. If the sign chart switches from positive to negative at a critical value, at *c*, it is a relative maximum. If the sign chart switches from negative to positive at a critical value, at *c*, it is a relative minimum.



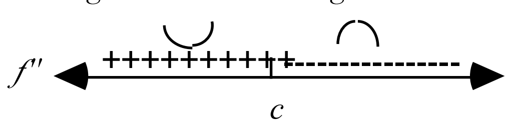
relative maximum



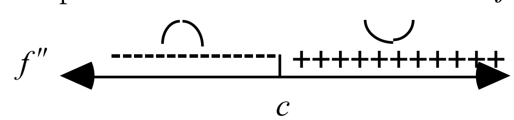
relative minimum

If there is no switch of signs at *c*, then *c* is not a relative minimum or maximum.

5. You have identified the *x*-values where extrema occur. If you are asked to find the **points** where relative maxima and minima occur, you must take every maximum and minimum value *c* found above and plug them into the function. That is, find $f(c)$.
6. Inflection points occur where the sign of the second derivative $f''(x)$ switches sign. Take the second derivative $f''(x)$, put in fraction form and find every value *c* where either the numerator or denominator of $f''(x)$ equals zero.
7. If the sign chart switches signs at *c* then an inflection point occurs at *c*. We don't care if $f''(c)$ exists.



Inflection point



Inflection point

If there is no switch of signs at *c*, then *c* is not a point of inflection.

8. You have identified the *x*-values where points of inflection occur. If you are asked to find the actual **point of inflection**, you must take every inflection point value *c* found above and plug it into the function. That is, find $f(c)$.

Example 5) Find all points of relative maximum and relative minimum and points of inflection if any. Justify your answers. Confirm by calculator.

a) $f(x) = x^3 - 3x^2$

$f'(x) = 3x^2 - 6x = 0$
$3x(x-2) = 0$
$(0,3)$ – rel max
$(2,-4)$ - rel min

$f''(x) = 6x - 6$
$(1,-2)$ - inf pt

b) $f(x) = 4x^3 - x^4$

$f'(x) = 12x^2 - 4x^3 = 0$
$4x^2(3-x) = 0$
$(3,27)$ – rel max

$f''(x) = 24x - 12x^2$
$12x(2-x) = 0$
$(0,0),(2,16)$ - inf pts

c) $f(x)=6x^5-10x^3$

$f'(x)=30x^4-30x^2=0$
$30x^2(x^2-1)=0$
$(1,-4)-$ rel min
$(-1,4)-$ rel max

$f''(x)=120x^3-60x$
$60x(2x^2-1)=0$
$(0,0),\left(\sqrt{\frac{1}{2}},-2.474\right)$ - inf
$\left(-\sqrt{\frac{1}{2}},2.474\right)$ - inf

d. $f(x)=-\cos x-\frac{1}{2}x$ on $[0,2\pi]$

$f'(x)=\sin x-\frac{1}{2}=0$
$\sin x=\frac{1}{2}$
$\left(\frac{5\pi}{6},\sqrt{\frac{3}{2}}-\frac{5\pi}{12}\right)-$ rel max
$\left(\frac{\pi}{6},-\sqrt{\frac{3}{2}}-\frac{\pi}{12}\right)-$ rel min

$f''(x)=\cos x$
$\cos x=0$
$\left(\frac{\pi}{2},-\frac{\pi}{4}\right),\left(\frac{3\pi}{2},-\frac{3\pi}{4}\right)$ - inf pt

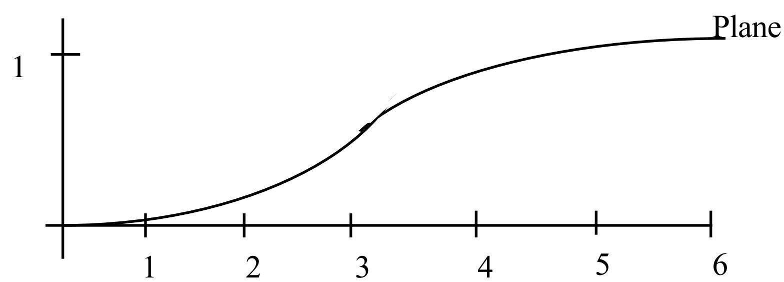
e. $f(x)=\frac{4}{x^2+4}$
Don't do inflection pts.

$f(x)=\frac{4}{x^2+4}$
$f'(x)=\frac{-8x}{(x^2+4)^2}=0$
$-8x=0\Rightarrow x=0$
$(0,1)-$ rel max

f. $f(x)=\frac{x^2+1}{x^2-9}$
Don't do inflection pts.

$f(x)=\frac{x^2+1}{x^2-9}$
$f'(x)=\frac{-20x}{(x^2-9)^2}=0$
$-20x=0\Rightarrow x=0$
$\left(0,\frac{-1}{9}\right)-$ rel max

Example 6) An airplane starts its descent when it is at an altitude of 1 mile, 6 miles east of the airport runway. Find the cubic function $f(x)=ax^3+bx^2+cx+d$ on the interval [0, 6] that describes a smooth glide path for the plane. Also find the location where the plane is descending at the fastest rate.



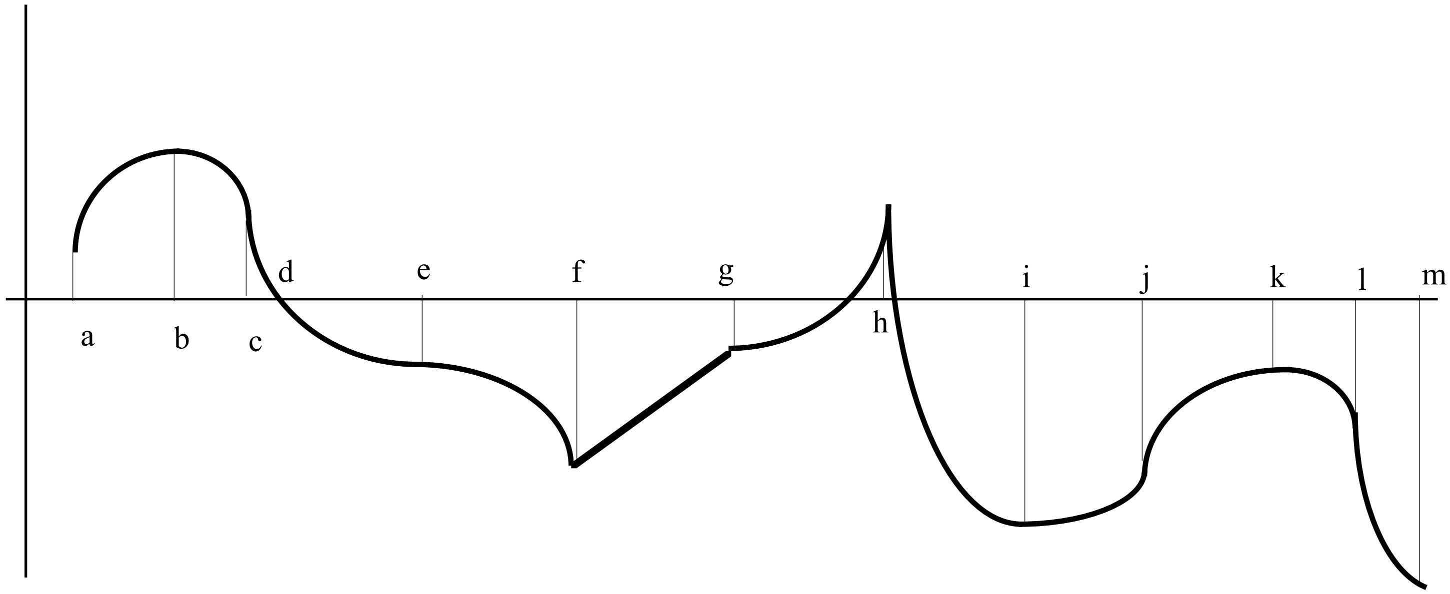
$f(x)=ax^3+bx^2+cx+d$
$f(x)=0+d=0\Rightarrow d=0$
$f(6)=216a+36b+6c=1$

$f'(x)=3ax^2+2bx+c$
$f'(0)=c=0$
$f'(6)=108a+12b+c=0$

$f(x)=\frac{-1}{108}x^3+\frac{1}{12}x^2$
$f'(x)=\frac{-1}{36}x^2+\frac{1}{6}x$
$f''(x)=\frac{-1}{18}x+\frac{1}{6}=0\Rightarrow x=3$ sec

$216a+36b=1,\ \ 108a+12b=0\Rightarrow a=\frac{-1}{108},b=\frac{1}{12}$

Function Analysis - Homework

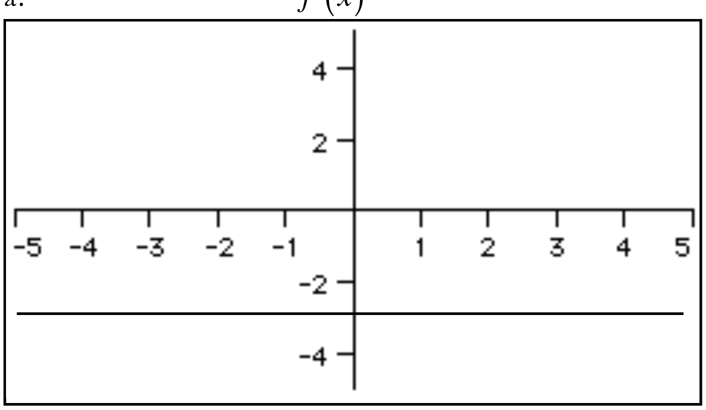


1. For each term, determine if it is applicable at the x-values a - m.

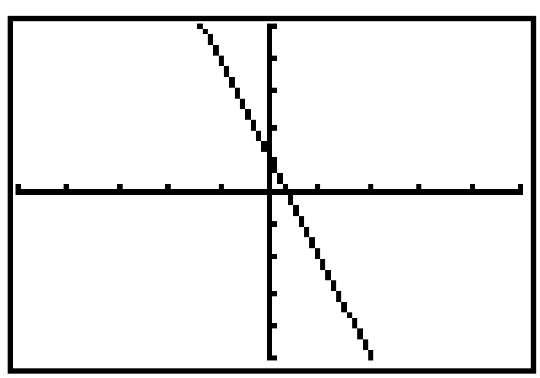
	Critical Point	Relative Minimum	Relative Maximum	Stationary Point	Inflection Point	Absolute Minimum	Absolute Maximum
a	x						
b	x		x	x			x
c	x				x		
d							
e	x			x	x		
f	x	x					
g	x						
h	x		x				
i	x	x		x			
j	x				x		
k	x		x	x			
l	x				x		
m						x	

2) You are given a graph of $f'(x)$. Draw a picture of a possible $f(x)$.

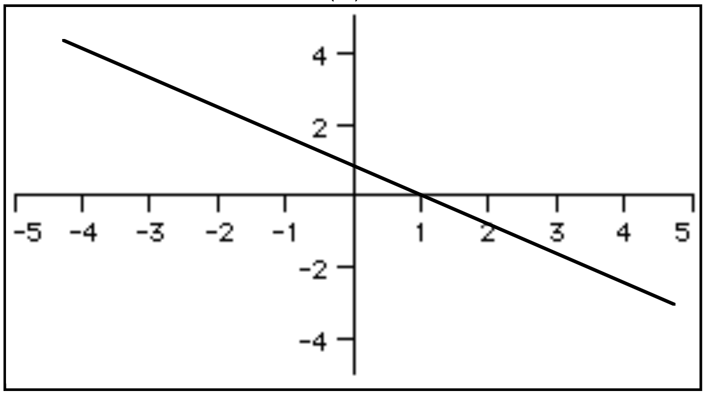
a. $f'(x)$

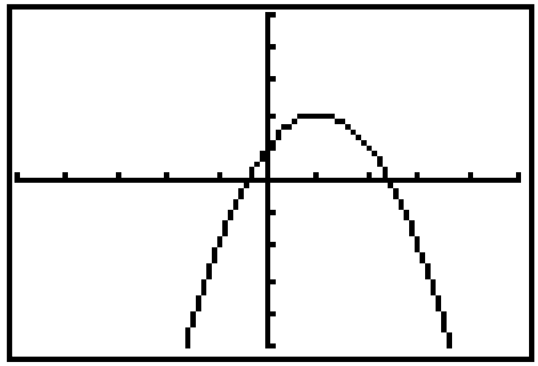


possible $f(x)$

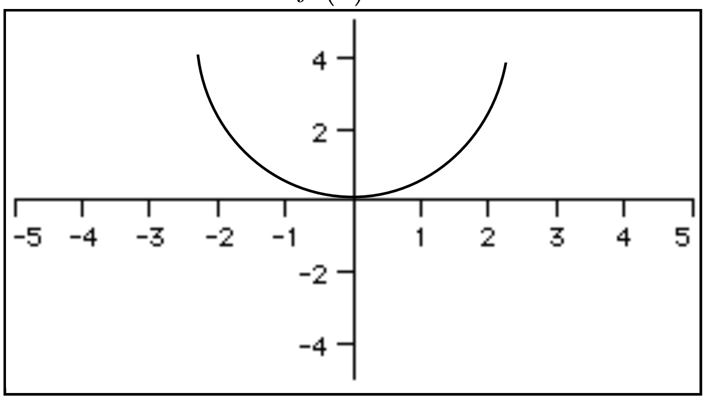


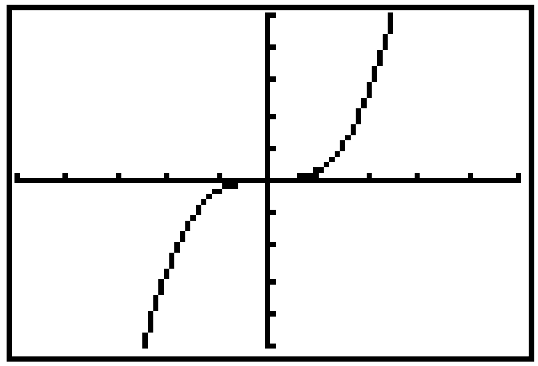
b. $f'(x)$ possible $f(x)$



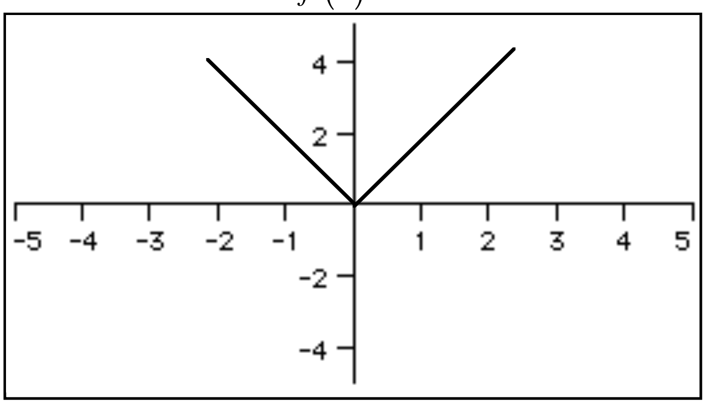


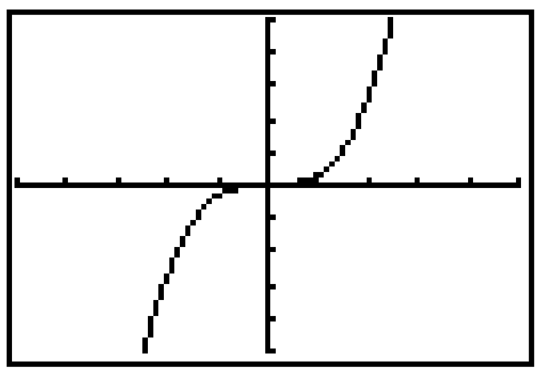
c. $f'(x)$ possible $f(x)$



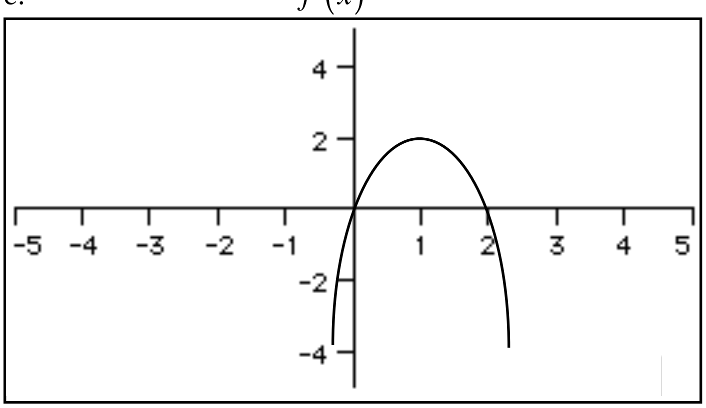


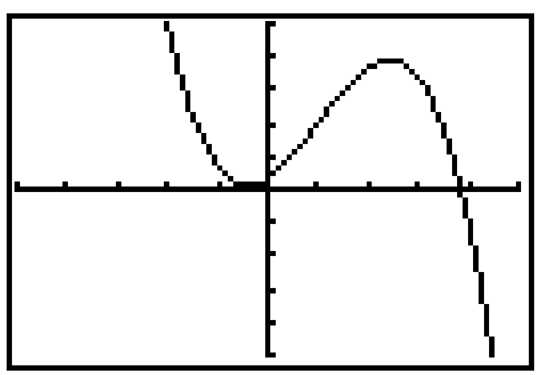
d. $f'(x)$ possible $f(x)$



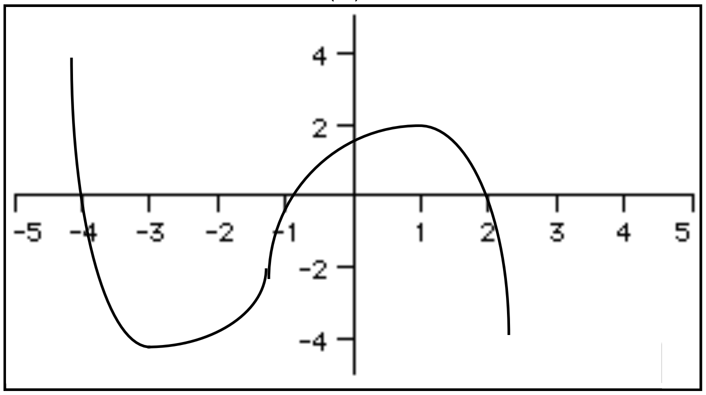


e. $f'(x)$ possible $f(x)$

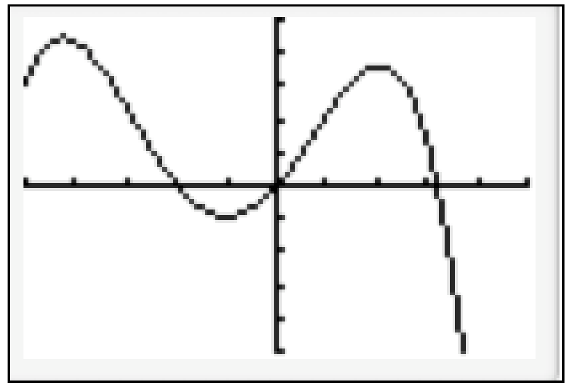




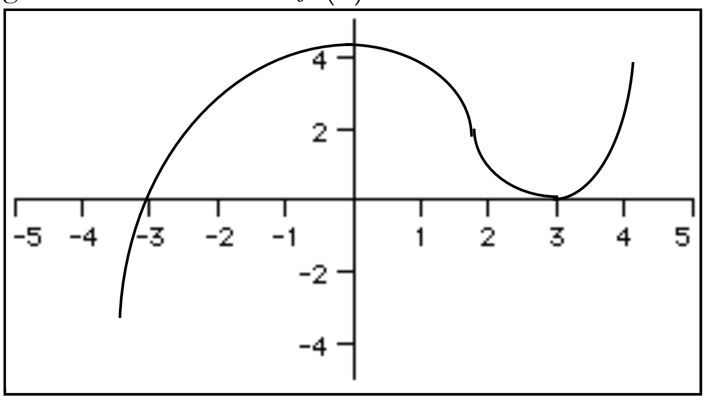
f. $f'(x)$



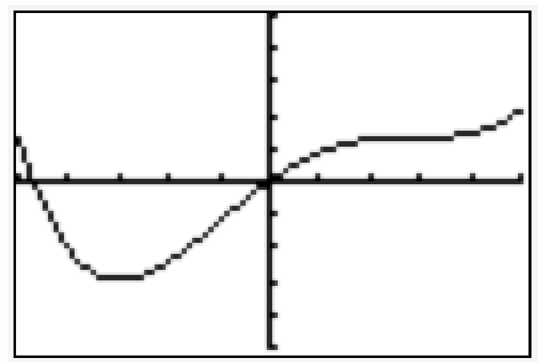
possible $f(x)$



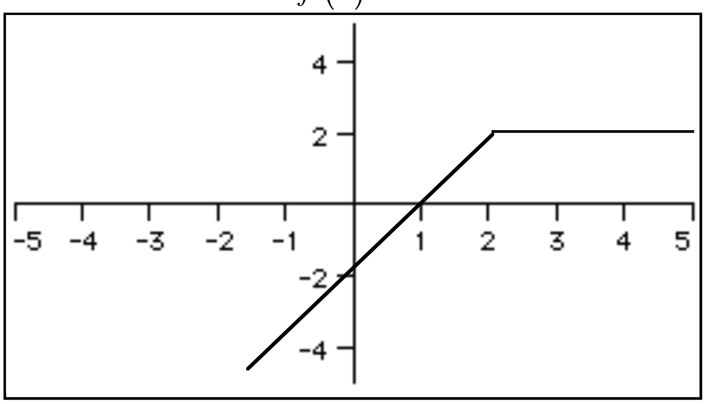
g. $f'(x)$



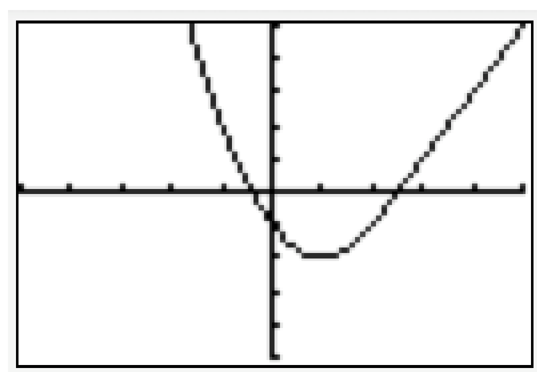
possible $f(x)$



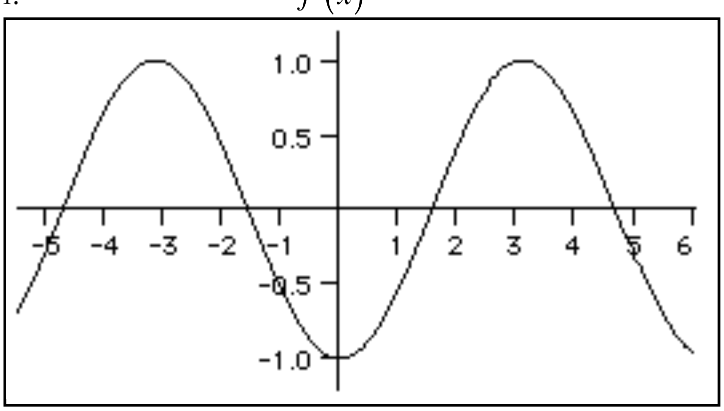
h. $f'(x)$



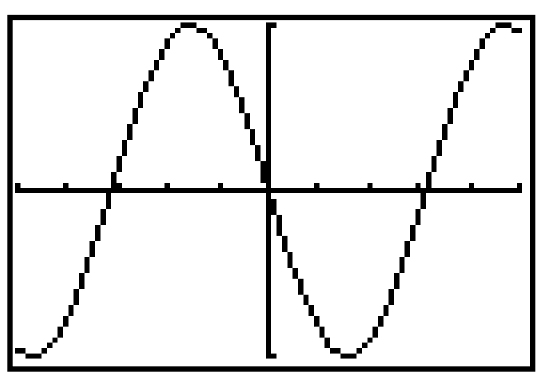
possible $f(x)$



i. $f'(x)$

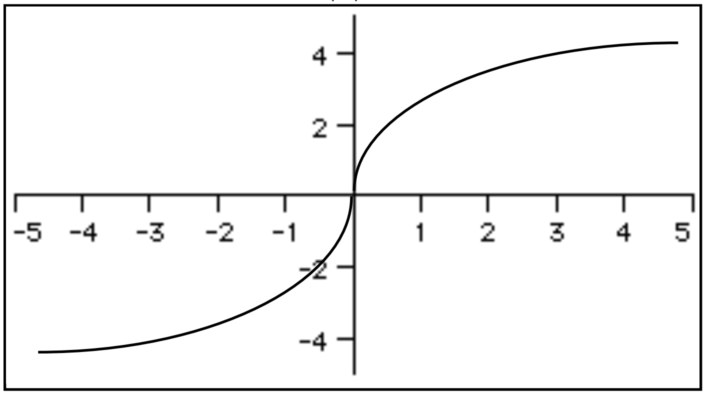


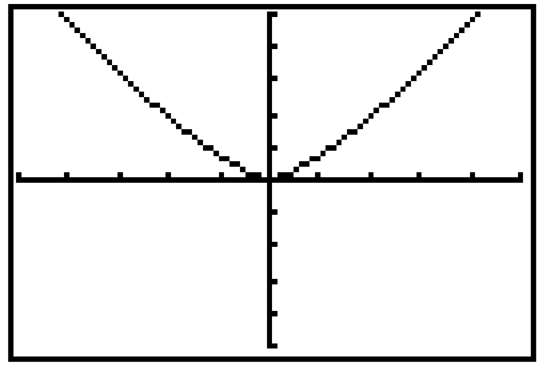
possible $f(x)$



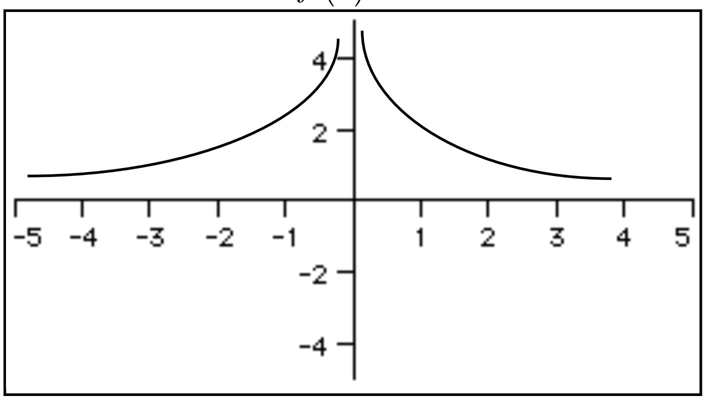
3) You are given a graph of $f'(x)$. Draw a picture of a possible $f(x)$.

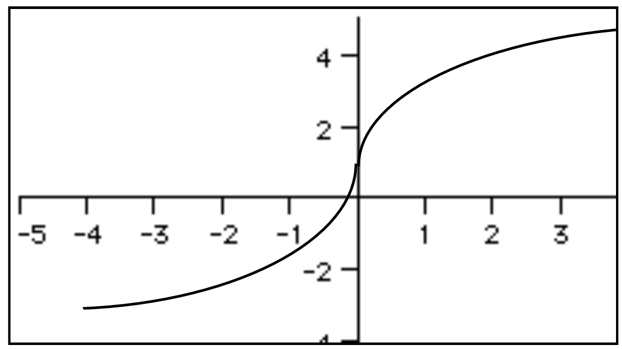
a. $f'(x)$ possible $f(x)$



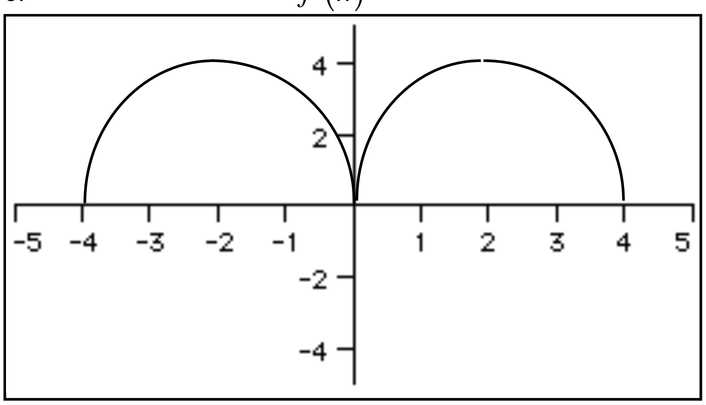


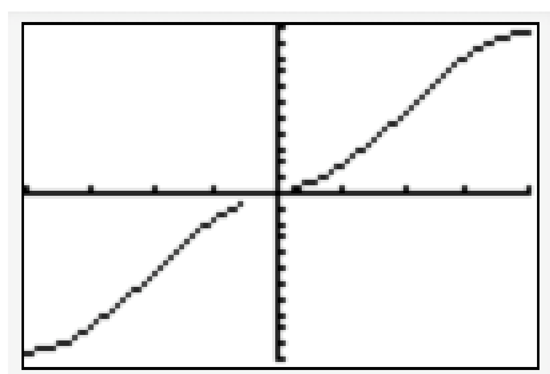
b. $f'(x)$ possible $f(x)$



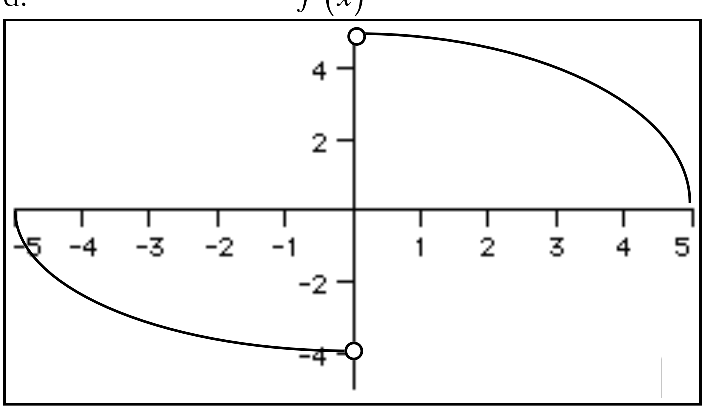


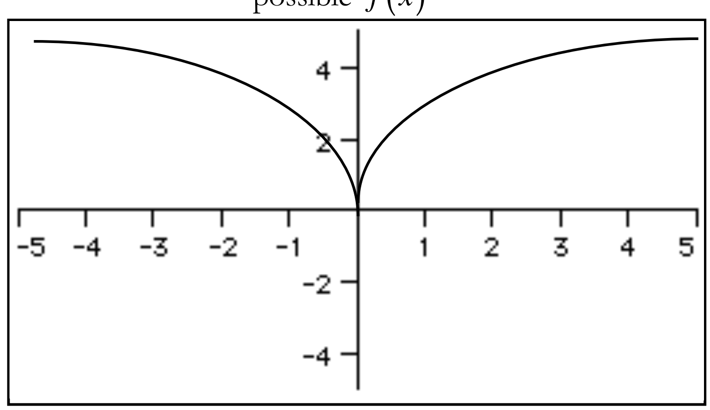
c. $f'(x)$ possible $f(x)$





d. $f'(x)$ possible $f(x)$

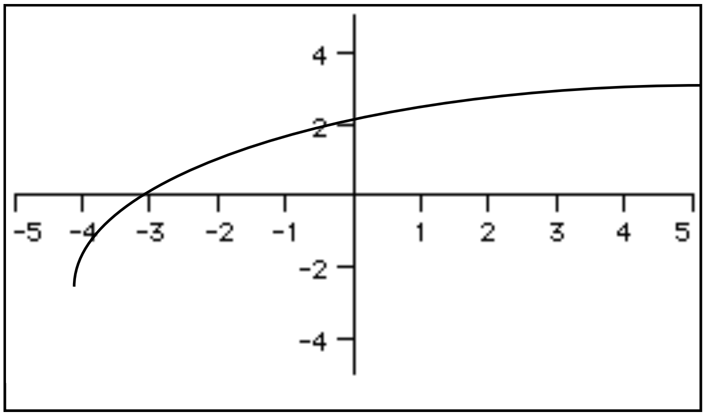


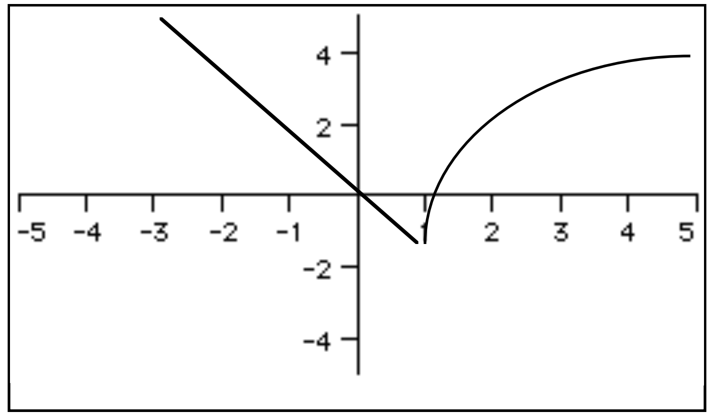


4) Sketch a possible $f(x)$ given the following information.

a. $f'(x) > 0, \quad f''(x) < 0$
$f(0) = 2$

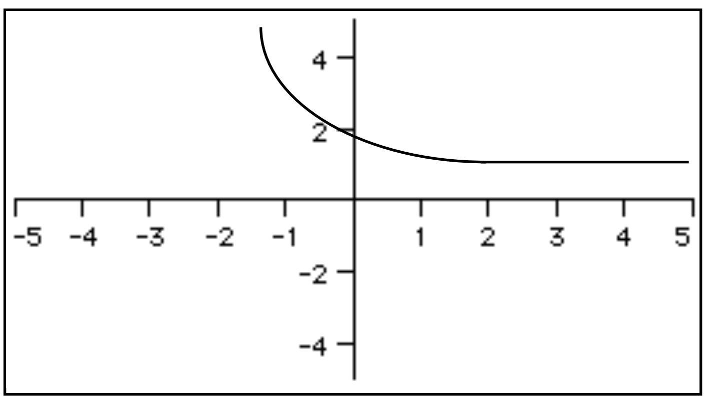
b. $f'(x) > 0, x > 1, \quad f'(x) = -1, x < 1$
$f(1) = -1 \quad \lim\limits_{x \to \infty} f(x) = 4$

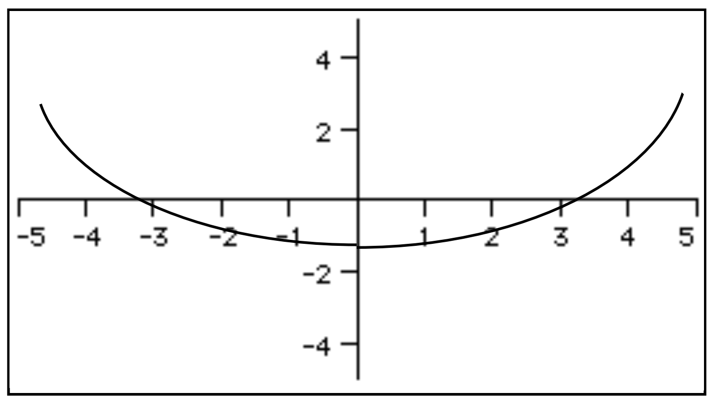




c. $f'(x) < 0, x < 2 \quad f''(x) > 0, x < 2$
$f(x) = 1, x \geq 2 \quad y - \text{intercept} = 2$

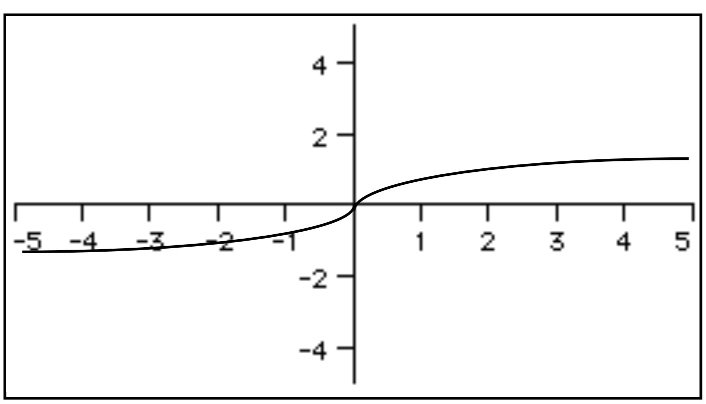
d. $f'(x) < 0, x < 0 \quad f'(x) > 0, x > 0$
$f''(x) > 0 \quad f(0) = -1$

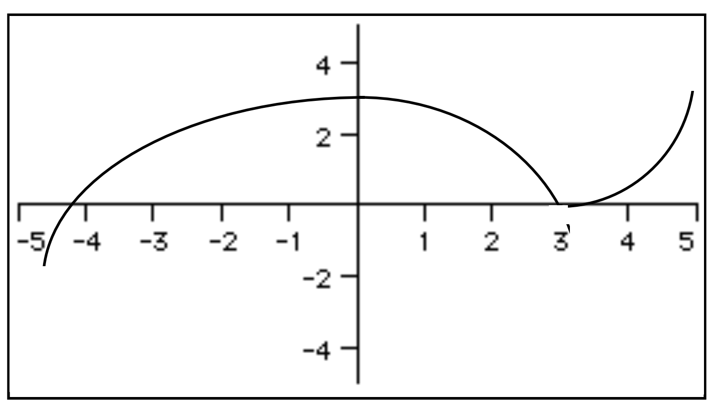




e. $f'(x) > 0 \quad f(0) = 0$
$\lim\limits_{x \to \infty} f(x) = 1 \quad \lim\limits_{x \to -\infty} f(x) = -1$

f. $f'(x) > 0, x < 0 \quad f'(x) > 0, x > 3 \quad f'(x) < 0, 0 < x < 3$
$f'(0) = 0 \quad f(0) = 3 \quad f(3) = 0$
$f''(x) < 0, x < 3 \quad f''(x) > 0, x > 3$

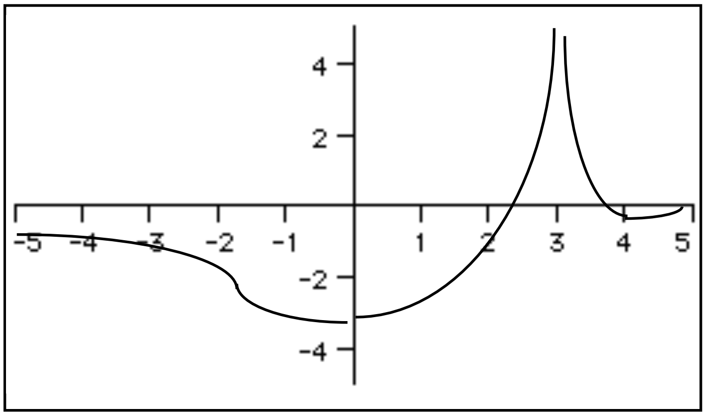




g. $f'(x)<0, x<0$ $\quad f'(x)<0, x>3$ $\quad f'(x)>0, 0<x<3$

$f'(0)=0$ $\quad f(0)=-2$ $\quad f''(-2)=0$

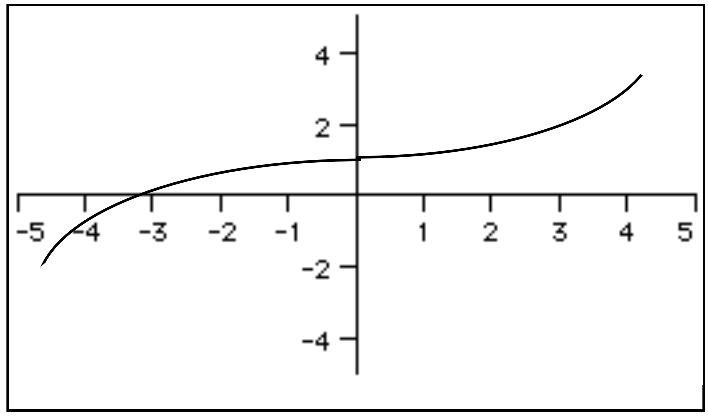
$\lim\limits_{x\to\pm\infty} f(x)=0$ $\quad \lim\limits_{x\to 3} f(x)=\infty$



h. $f'(x)>0, x\neq 0$ $\quad f'(0)=0$

$f''(x)<0, x<0$ $\quad f''(x)>0, x>0$

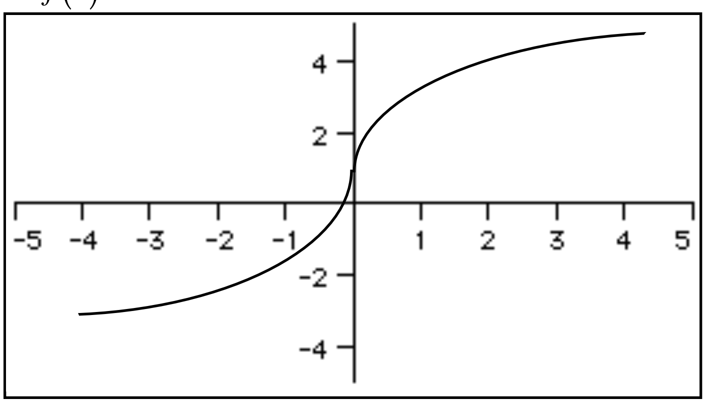
$f(0)=1$



i. $f'(x)>0, x\neq 0$ $\quad f'(0)$ DNE

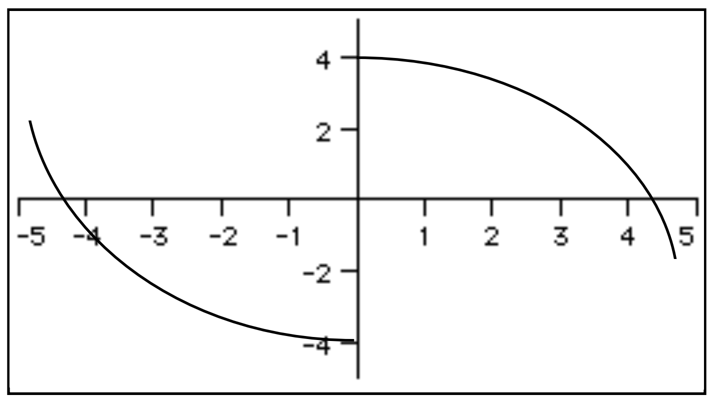
$f''(x)>0, x<0$ $\quad f''(x)<0, x>0$

$f(0)=1$



j. $f'(x)<0, x>0$ $\quad f''(x)<0, x>0$

$\lim\limits_{x\to 0^+} f(x)=4$ $\quad f$ is symmetric to the origin



5) Find all points of relative maximum and relative minimum and points of inflection if any. Justify your answers. Confirm by calculator.

a. $f(x)=x^2-8x+4$

$f'(x)=2x-8$

$2x-8=0 \Rightarrow x=4$

$(4,-12)$ - rel min

$f''(x)=2>0$

no inflection pts

b. $f(x)=1+12x-3x^2-2x^3$

$f'(x)=12-6x-6x^2$

$-6(x^2+x-2)=0 \Rightarrow x=1,-2$

$(-2,-19)$ - rel min

$(1,8)$ - rel max

$f''(x)=-6-12x$

$6x=12 \Rightarrow x=-.5$

$(-.5,-5.5)$ - infl. pt.

c. $f(x)=(2x-5)^3$

$f'(x)=6(2x-5)^2$

$6(2x-5)^2=0 \Rightarrow x=\dfrac{5}{2}$

No extrema

$f''(x)=24(2x-5)^2$

$24(2x-5)^2=0 \Rightarrow x=\dfrac{5}{2}$

$\left(\dfrac{5}{2},0\right)$ - infl. pt.

d. $f(x) = 3\sqrt[3]{x} - 2$

$$f'(x) = \frac{1}{x^{2/3}}$$
$x^{2/3} = 0 \Rightarrow x = 0$
No extrema
$$f''(x) = \frac{2}{3x^{5/3}}$$
$3x^{5/3} = 0 \Rightarrow x = 0$
$(0,-2)$ - infl. pt.

e. $f(x) = \dfrac{x^2}{x^2 - 4}$ (don't do inflection pts)

$$f'(x) = \frac{-8x}{\left(x^2 - 4\right)^2}$$
$-8x = 0 \Rightarrow x = 0$
$(0,0)$ - rel max
Don't do inflection pts

f. $f(x) = \sin^2 x + \sin x \quad [0,2\pi]$
(don't do inflection pts)

$f'(x) = 2\sin x \cos x + \cos x$
$$\cos x(2\sin x + 1) = 0 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$$
$\left(\frac{\pi}{2}, 2\right), \left(\frac{3\pi}{2}, 0\right)$ - rel max
$\left(\frac{7\pi}{6}, -\frac{1}{4}\right), \left(\frac{11\pi}{6}, -\frac{1}{4}\right)$ - rel min

g. $f(x) = x - \cos x \quad [0,2\pi]$

$f'(x) = 1 + \sin x$
$$1 + \sin x = 0 \Rightarrow x = \frac{3\pi}{2}, \frac{7\pi}{2}$$
No relative extrema
$f''(x) = \cos x$
$$\cos = 0 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$$
$\left(\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$ - infl. pt.

h. $f(x) = x\sqrt{x + 1}$ (don't do inflection pts)

$$f'(x) = \frac{3x + 2}{x\sqrt{x + 1}}$$
$$3x + 2 = 0 \Rightarrow x = \frac{-2}{3}$$
$\left(\frac{-2}{3}, \frac{-2}{3}\sqrt{\frac{1}{3}}\right) = \left(\frac{-2}{3}, -.385\right)$, rel min

i. $f(x) = \left(x^2 - 16\right)^{2/3}$ (don't do inflection pts)

$f(x) = \left(x^2 - 16\right)^{2/3}$
$$f'(x) = \frac{4x}{3\left(x^2 - 16\right)^{1/3}}$$
$4x = 0 \Rightarrow x = 0$
$\left(x^2 - 16\right) = 0 \Rightarrow x = \pm 4$
$\left(0, (-16)^{2/3}\right) = (0, 6.346)$ - rel max
$(4,0), (-4,0)$ - rel min

Finding Absolute Maximums and Minimums - Classwork

Suppose you were asked to find the student in this school who, at this point in time, has the most money on him or her. Write below a method that would be efficient and quick.

Find the maximum in each room, send each to a common location and find the max of the max.

Our goal will be to find a maximum value of a function and minimum value of a function on a closed interval. Knowing that relative maxima and minima occur only at critical values (where the derivative equals zero or fails to exist), the method for finding absolute maxima and minima on a closed interval $[a,b]$ is as follows:

1. Find the critical value of f in (a,b). (set both the numerator and denominator of $f'=0$ and solve).
2. Evaluate f at each critical number in (a,b).
3. Evaluate f at each endpoint of $[a,b]$ - that is find $f(a)$ and $f(b)$.
4. The smallest of these is the absolute minimum. The largest of these is the absolute maximum.
5. Remember to discern the difference between **where** the absolute min or max occurs as opposed to **what** the absolute min or max is. **Where** is the *x*-value. **What** is the *y*-value.

Examples) Find the absolute minimum and maximum values of the following functions. Justify your answers.

a. $f(x)=3x^2-24x-1 \quad [-1,5]$

$f'(x)=6x-24$
$6x=24 \Rightarrow x=4$
$f(4)=-49$ - Abs min
$f(-1)=26$ - Abs max
$f(5)=-46$

b. $f(x)=6x^3-6x^4+5 \quad [-1,2]$

$f'(x)=18x^2-24x^3$
$6x(3-4x)=0 \Rightarrow x=0,\dfrac{3}{4}$
$f(0)=5$
$f\left(\dfrac{3}{4}\right)=5.633$ - Abs max
$f(-1)=-7$
$f(2)=-43$ - Abs min

c. $f(x)=3x^{2/3}-2x+1 \quad [-1,8]$

$f'(x)=\dfrac{2}{x^{1/3}}-2$
$\dfrac{2-2x^{1/3}}{x^{1/3}}=0 \Rightarrow x=1$
$x=0$ (where y' DNE)

$f(0)=1$
$f(1)=2$
$f(-1)=6$ - Abs max
$f(8)=-3$ - Abs min

d. $f(x)=\sin^2 x+\cos x \quad [0,2\pi]$

$f'(x)=2\sin x\cos x-\sin x$
$\sin x(2\cos x-1)=0 \Rightarrow x=0,\pi,2\pi,\dfrac{\pi}{3},\dfrac{5\pi}{3}$
$f(0)=1$
$f(\pi)=-1$ - Abs min
$f\left(\dfrac{\pi}{3}\right)=\dfrac{5}{4}$ - Abs max
$f\left(\dfrac{5\pi}{3}\right)=\dfrac{5}{4}$ - Abs max

Finding Absolute Maximums and Minimums - Homework

Find the absolute maximums and minimums of f on the given closed interval and state where these values occur.

1. $f(x) = 4x^2 - 4x + 1$ [0, 2]

$f'(x) = 8x - 4$
$4(2x-1) = 0 \Rightarrow x = \frac{1}{2}$

$f\left(\frac{1}{2}\right) = 0$ - Abs min
$f(0) = 1$
$f(2) = 9$ - Abs max

2. $f(x) = 2x^3 - 3x^2 - 12x - 1$ [-2, 3]

$f'(x) = 6x^2 - 6x - 12$
$6(x^2 - x - 2) = 0 \Rightarrow x = -1, 2$

$f(-1) = 6$ - Abs max
$f(2) = -21$ - Abs min
$f(-2) = -5$
$f(3) = -10$

3. $f(x) = \dfrac{x}{x^2 + 2}$ [-1, 4]

$f'(x) = \dfrac{2 - x^2}{(x^2+2)^2}$
$2 - x = 0 \Rightarrow x = \pm\sqrt{2}$

$f(\sqrt{2}) = .354$ - Abs max
$f(-1) = -.333$ - Abs min
$f(4) = .222$

4. $f(x) = (x^2 - 2)^{2/3}$ [-2, 3]

$f'(x) = \dfrac{4x}{3(x^2-2)^{1/3}}$
$4x = 0 \Rightarrow x = 0$
$(x^2 - 2) = 0 \Rightarrow x = \pm\sqrt{2}$

$f(0) = 1.587$
$f(\sqrt{2}) = 0$ - Abs min
$f(-\sqrt{2}) = 0$ - Abs min
$f(3) = 3.659$ - Abs max

5. $f(x) = x^{2/3}(20 - x)$ [-1, 20]

$f'(x) = \dfrac{-5(x-8)}{3x^{1/3}}$
$x - 8 = 0 \Rightarrow x = 8$
$3x^{1/3} = 0 \Rightarrow x = 0$

$f(8) = 48$ - Abs max
$f(0) = 0$ - Abs min
$f(-1) = 21$
$f(20) = 0$ - Abs min

6. $f(x) = \sin x - \cos x$ [0, π]

$f'(x) = \cos x + \sin x$
$\cos x = -\sin x$
$\tan x = -1 \Rightarrow x = \frac{3\pi}{4}$

$f\left(\frac{3\pi}{4}\right) = \sqrt{2}$ - Abs max
$f(0) = -1$ - Abs min
$f(\pi) = 1$

7. $f(x) = x - \tan x$ $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

$f'(x) = 1 - \sec^2 x$
$1 = \dfrac{1}{\cos^2 x} \Rightarrow x = 0$

$f(0) = 0$
$f\left(\frac{-\pi}{4}\right) = .215$ - Abs max
$f\left(\frac{\pi}{4}\right) = -.215$ - Abs min

8. $f(x) = |6 - 4x|$ [-3, 3]

$6 - 4x = 0 \Rightarrow x = \frac{3}{2}$
$f\left(\frac{3}{2}\right) = 0$ - Abs min

$f(-3) = 18$ - Abs max
$f(3) = 6$

9. What is the smallest possible slope to $y = x^3 - 3x^2 + 5x - 1$

$y = x^3 - 3x^2 + 5x - 1$
$y' = 3x^2 - 6x + 5$
$y'' = 6x - 6 = 0 \Rightarrow x = 1$
$y'(1) = 2$

10. If a particle moves along a straight line according to $s(t) = t^4 - 4t^3 + 6t^2 - 20$, find
a) the maximum & minimum velocity on $0 \le t \le 3$.
b) the maximum & minimum acceleration on $0 \le t \le 3$.

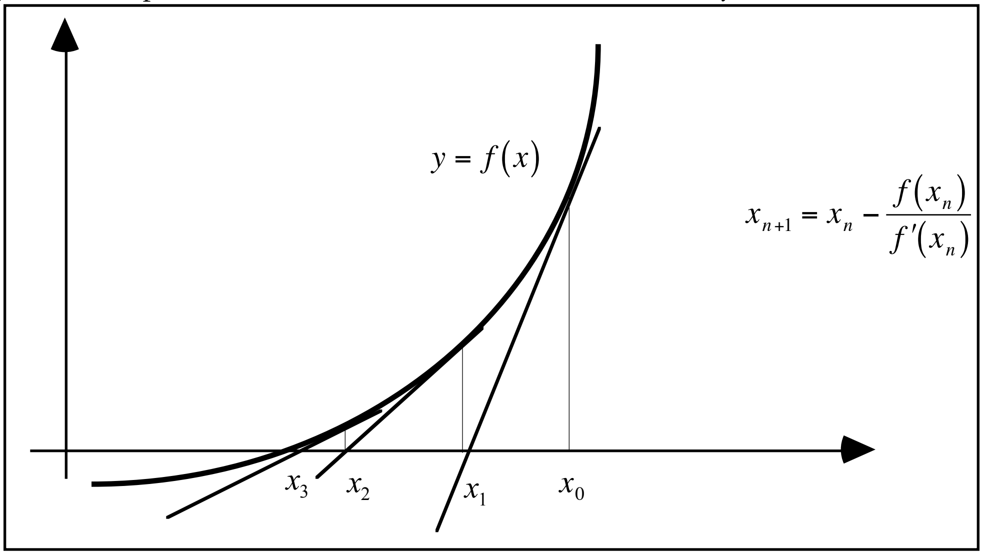
$s(t) = t^4 - 4t^3 + 6t^2 - 20$
$v(t) = 4t^3 - 12t^2 + 12t$
$v'(t) = 12t^2 - 24t + 12$

$v'(t) = 12(t^2 - 2t + 1)$
$12(t^2 - 2t + 1) = 0 \Rightarrow t = 1$
$v(1) = 4, v(0) = 0, v(3) = 36$

$a(t) = 12t^2 - 24t + 12$
$a'(t) = 24t - 24 \Rightarrow t = 1$
$a(1) = 0, a(0) = 12, a(3) = 48$

Newton's Method of Roots - Classwork

The concept of Newton's method of finding roots is based on making an initial guess x_0 of the root of the function $f(x)$. From there, you come up with a series of x_i , which will be successively closer to the root of $f(x)$.



Sample problems - do by hand. You may use your calculators only for basic operations. Complete two iterations of Newton's method using the indicated initial guess.

1. $f(x) = x^2 - 2 \qquad x_1 = 1$

x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$
1	-1	2	$\frac{-1}{2}$
1.5	$\frac{1}{4}$	3	$\frac{1}{12}$
$\frac{17}{12}$			

2. $f(x) = x^3 - x^2 - 2x - 2 \qquad x_1 = 2$

x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$
2	-2	6	$\frac{-1}{3}$
$\frac{7}{3}$	$\frac{16}{27}$	$\frac{29}{3}$	$\frac{16}{261}$
$\frac{593}{261}$			

1. On all calculators, place the function which you wish the solution to in Y1.
2. Place the derivative of the function in Y2.
3. It is best to set your MODE to FLOAT giving the maximum amount of accuracy. However, if you are asked to apply Newton until two approximations differ by less than d decimal places, you may wish to set your MODE to d decimal places first.
4. Make your initial Guess and store it as X. Then type X-Y1/Y2 STO X. You will get the first iteration.
5. Press ENTER and you will get successive iterations.
6. To do another problem, simply repeat the steps above. You can use 2nd ENTER so you do not have to type the command on line 4.

3. Find the roots of $f(x) = 5x^2 - 4x - 7$

$\boxed{1.649 \text{ and } -0.849}$

4. Find the smallest positive root of $f(x) = \sin x - x^2 + 1$

$\boxed{1.410}$

Warning: Do not depend on this routine on the A.P. exam. They usually give you a problem to calculate a root of a function through one or two iterations where calculators are not allowed. You must know the formula and how to apply it algebraically. Newton's method fails if the derivative of the function is zero at your original guess. Note: Newton is no longer on the A.P. exam.

Newton's Method of Roots - Homework

In the following exercises, use Newton's Method by hand to find the first iteration in approximating the zeros and continue the process with the calculator until 2 successive approximations differ by less than .001.

1. $f(x) = x^3 + x + 3$ (Initial Guess = 1)

x	$f(x)$	$f'(x)$	$f(x)/f'(x)$
1.000	5.0000	4.0000	1.2500
-0.250	2.7344	1.1875	2.3026
-2.553	-16.1854	20.5478	-0.7877
…	…	…	…
-1.213	0.0000	5.4171	0.0000

2. $f(x) = x^5 + x + 3$ (Initial Guess = 1)

x	$f(x)$	$f'(x)$	$f(x)/f'(x)$
1.000	5.0000	6.0000	0.8333
0.167	3.1668	1.0039	3.1546
-2.988	-238.1499	399.5364	-0.5961
…	…	…	…
-1.133	0.0000	9.2392	0.0000

3. $f(x) = x^2 - \dfrac{1}{x-1}$ (Initial Guess = 3)

x	$f(x)$	$f'(x)$	$f(x)/f'(x)$
3.000	8.5000	6.2500	1.3600
1.640	1.1271	5.7214	0.1970
1.443	-0.1751	7.9815	-0.0219
1.465	-0.0048	7.5559	-0.0006
1.466	0.0000	7.5446	0.0000

4. $f(x) = x^4 - 10x^2 - 7$ (Initial Guess = 3)

x	$f(x)$	$f'(x)$	$f(x)/f'(x)$
3.000	-16.0000	48.0000	-0.3333
3.333	5.3457	81.4815	0.0656
3.268	0.2402	74.2172	0.0032
3.264	0.0006	73.8677	0.0000

5. $f(x) = 2x + \sin(x+1)$ (Initial Guess = $-\dfrac{\pi}{6}$)

x	$f(x)$	$f'(x)$	$f(x)/f'(x)$
-0.524	-0.5886	2.8887	-0.2038
-0.320	-0.0107	2.7775	-0.0039
-0.316	0.0000	2.7750	0.0000

6. $f(x) = x^3 - \cos x + 2$ (Initial Guess = 3)

x	$f(x)$	$f'(x)$	$f(x)/f'(x)$
3.000	29.9900	27.1411	1.1050
1.895	9.1240	11.7214	0.7784
1.117	2.9536	4.6392	0.6367
…	…	…	…
-1.173	0.0000	3.2031	0.0000

7. $f(x) = \sin x$ has a root at $x = 0$. Suppose your initial guess is 1.3. Show that Newton's method does not work here and formulate a reason why it does not. Look at the picture visually in order to understand what is happening.

x	$f(x)$	$f'(x)$	$f(x)/f'(x)$
1.300	0.9636	0.2675	3.6021
-2.302	-0.7443	-0.6678	1.1145
-3.417	0.2715	-0.9624	-0.2821
-3.134	-0.0071	-1.0000	0.0071
-3.142	0.0000	-1.0000	0.0000

you get further away each iterative.

8) Newton's method can be used to determine square roots. For $x = \sqrt{a}$, use the equation $f(x) = x^2 - a$. Use this method to find $\sqrt{7}$ and $\sqrt{214}$. Use this method to find $\sqrt[5]{5}$

$\sqrt{7} = 2.646 \qquad \sqrt{214} = 14.629 \qquad \sqrt[5]{5} = 1.380$

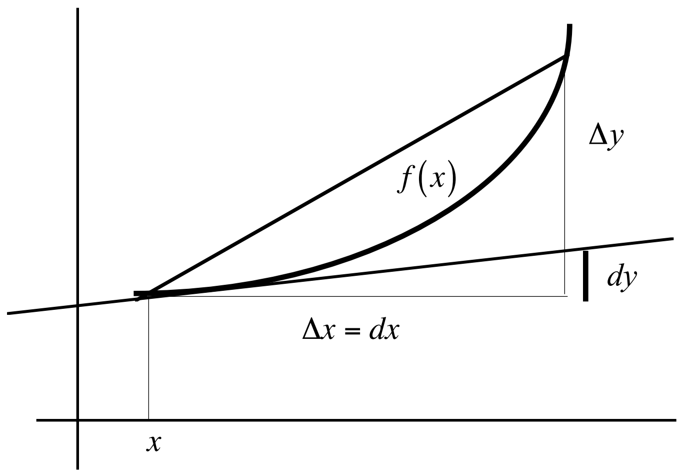
9. Show that Newton's method fails to converge for the function $f(x) = x^{1/3}$ using $x_1 = .1$

$f(x)$ is not differentiable at $x = 0$. $\quad f(x) = x^{1/3} \quad f'(x) = \dfrac{1}{3x^{2/3}}$

$x_2 = -.2, x_3 = .4, x_4 = -.8, \ldots$

Approximation using Differentials - Classwork

We've defined $\frac{dy}{dx}$ to be $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$. In this section, we give separate meanings to Δx, Δy, dx, dy.



As x changes along the curve $y = f(x)$, y changes as well. We call Δx and dx the change in x. $\Delta x = dx$.
Δy is the actual change in y given Δx (the secant line). $\Delta y = f(x + \Delta x) - f(x)$
dy is the approximate change in y (the tangent line). As $\Delta x = dx$ gets very small, Δy and dy are very close in value. We are usually interested in Δy, the actual change in y. But without technology, Δy can be difficult to calculate.
dy is easier. $\frac{dy}{dx} = f'(x)$. So it follows that $dy = f'(x) \cdot dx$. This is called the differential form.

Example 1: $y = f(x) = 3x^2 - 8x + 2$ Find Δy and dy when $x = 1$ and $\Delta x = dx = .01$.

$\Delta y = f(x + \Delta x) - f(x)$
$\Delta y = 3(x + \Delta x)^2 - 8(x + \Delta x) + 2 - 3x^2 + 8x - 2$
$\Delta y = 3(1.01)^2 - 8(1.01) + 2 - 3 + 8 - 2$
$\Delta y = -.0197$

$dy = f'(x) \cdot dx$
$f'(x) = 6x - 8$
$dy = -2(.01) = .-02$

Example 2: Compare Δy and dy as x changes from 2 to 2.01 given $f(x) = x^3 - x^2 + 3x - 2$

$\Delta y = f(x + \Delta x) - f(x)$
$\Delta y = (2.01)^3 - (2.01)^2 + 3(2.01) - 2 - 8$
$\Delta y = .1105$

$dy = f'(x) \cdot dx$
$f'(x) = 3x^2 - 2x + 3$
$dy = 11(.01) = .11$

Example 3) Find the approximate error in calculating the volume of a sphere if the radius is measured to be 5 inches with a measurement error of ± 0.1 inches. Find the relative percentage error $\frac{dv}{V}$ as well.

$V = \frac{4}{3}\pi r^3$
$dV = 4\pi r^2 dt$
$dV = 4\pi(25)(\pm.1) = 10\pi = 31.416 \text{ in}^3$

$\frac{dv}{V} = \frac{10\pi}{\frac{500\pi}{3}} = \frac{3}{50} = 6\%$

Approximation using Differentials - Homework

Use differentials to compute Δy and dy given $f(x)$, x, and $\Delta x = dx$.

1. $y = 5x^2 - 9x - 5$

   $x = 2,\ \Delta x = dx = .01$

   $\boxed{\begin{array}{l} \Delta y = .1105 \\ dy = 11(.01) = .11 \end{array}}$

2. $f(x) = 2x^3 - 4x^2 - 6x + 5$

   $x = 1,\ \Delta x = dx = .01$

   $\boxed{\begin{array}{l} \Delta y = -.0798 \\ dy = -8(.01) = -.08 \end{array}}$

3. $f(x) = \dfrac{1}{x^2}$

   $x = 2,\ \Delta x = dx = .01$

   $\boxed{\begin{array}{l} \Delta y = -.00248 \\ dy = -8(.01) = -.0025 \end{array}}$

4. The side of a square is measured to be 10 feet with a possible error of ± 0.1 ft. What is the difference between the true greatest possible error in computing the area of the square and its approximation using differentials. Find the relative percentage error between the approximate error and true area.

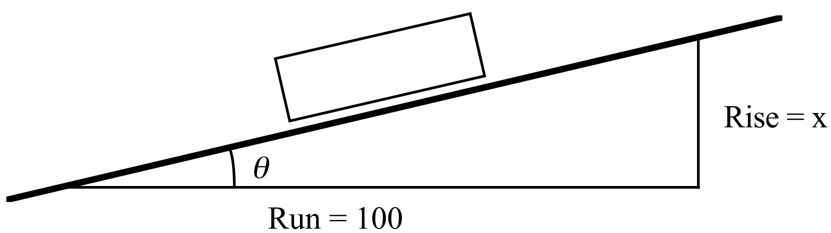
$\boxed{\begin{array}{l} \Delta A = .2001 \text{ in}^2 \\ dA = 20(.01) = .2 \text{ in}^2 \end{array}}$ $\boxed{\text{Difference } = \ .0001 \text{ in}^2}$ $\boxed{\% \text{ Error} = \dfrac{.2}{100} = .2\%}$

5. An 8 inch tall cylindrical coffee can has a 6 inch diameter with a possible error of ..02 inch. What is the difference between the true greatest possible error in computing the volume of the can and its approximation using differentials. Find the relative percentage error between the approximate error and true volume.

$\boxed{\begin{array}{l} \Delta V = \pi(3.01)^2(8.02) - \pi(9)(8) = 2.0797 \text{ in}^3 \\ dV = \pi\left[\left(r^2\right)dh + 2r \cdot h \cdot dr\right] \\ dV = \pi\left[\left(3^2\right)(.02) + 2(3)(8)(.01)\right] = 2.0735 \text{ in}^3 \end{array}}$ $\boxed{\% \text{ Error} = \dfrac{2.0735}{226.1847} = .9\%}$

6. On roads in hilly areas, you sometimes see signs like this:

Steep hill
20% Grade



The grade of a hill is the slope (rise/run) written as a percentage, or, equivalently, the number of feet the hill rises per hundred feet horizontally. The figure above shows the latter meaning of grade.

a) Let x be the grade of a hill. Explain why the angle, θ degrees, that a hill makes with the horizontal is given by $\theta = \dfrac{180}{\pi}\tan^{-1}\dfrac{x}{100}$ - $\boxed{\text{changes the angle in radians to degrees}}$

b) An equation for $d\theta$ is $d\theta = \left(\dfrac{180}{\pi}\right)\dfrac{100}{x^2 + 10000}dx$. You can estimate θ at x = 20% simply by multiplying $d\theta$ at x = 0 by 20. How much error is there in the value of θ found by this way rather than using the exact formula which involves the $\tan^{-1}$ function?

$$\boxed{d\theta = \left(\frac{180}{\pi}\right)\frac{100}{x^2 + 10000}(20) = 11.459° \quad \theta = \frac{180}{\pi}\tan^{-1}\frac{20}{100} = 11.310°}$$

c) A rule of thumb you can use to estimate the number of degrees a hill makes with the horizontal is to divide the grade by 2. When you use this method to determine the the number of degrees for a 20 % grade, how much error is there in the number? $\boxed{1.310°}$

Optimization Problems - Classwork

Many times in life we are asked to do an optimization problem - that is, find the largest or smallest value of some quantity that will fufill a need. Typical situations are:

- find the route which will minimize the time it takes me to get to school.
- build a structure using the least amount of material.
- build a structure costing the least amount of money.
- build a yard enclosing the most amount of space.
- find the least medication one should take to help a medical problem.
- find how the most one should charge for a CD in order to make as much money as possible.

All of these situations have something in common - they are all trying to maximize or minimize some quantity. This lends itself to a calculus solution. We have spent the better part of last month trying to find maximum and minimum values of functions. In every optimization problem, you are always looking for a quantity to be maximimed or minimized. So in solving word problems, you must look carefully for certain words among all the verbiage. Look for words like “minimize area”, “smallest volume”, “least amount of time”, “shortest distance”, “cheapest price.” On the following pages, there are a wealth of problems. Quickly examine each and underline the key words which tell you what kind of problem it is.

Methods for Solving Optimization Problems

1. Assign variables to all given quantities and quantities to be determined. Don’t be afraid to use letters you usually do not use (*p*,*m*,*g*, etc.). When feasible, make a sketch of the problem.

2. Making a chart of possible answers allows you to see a relationship between variables. While not necessary, it is helpful.

3. Write a “primary” equation for the quantity you found that needs to be maximized or minimized

   Area of Rectangle = length • width — Hypotenuse = $\sqrt{x^2 + y^2}$
   Distance = rate • time — Perimeter of rectangle = 2• length + 2• width
   Volume of rectangular solid = length • width • height — Volume of cylinder = π(radius^2) • height

4. Reduce the right side of this “primary equation” to one having a single variable. If there is more than one variable on the right side, you must write a “secondary” equation (a restriction or constraint) relating the variables of the primary equation.

5. Take the derivative of the equation and **set equal to zero**. If you get more than one answer, make a sign chart to determine whether it represents a maximum or minimum. Pay attention to whether that value makes sense. Time is rarely negative (it can’t take negative 7 hours to run a race). You cannot use more than you have (you can’t have a length of 8 feet when you only have 6 feet of fencing).

6. **Be sure that you answer the question that is asked.** If you are asked to find a minimum or maximum value of some quantity, you must plug your answer from (4) into your primary equation.

7. If you are to find a maximum or minimum on a closed interval, you must test the endpoints as well. Make sure your work is clear.

8. You can verify your answers by graphing your primary equation with one variable on the calculator. Use your $\boxed{\text{2nd}}$ $\boxed{\text{CALC}}$ maximum or minimum function.

Example 1) Two numbers add up to 40. Find the numbers that maximize their product.

Smaller Number	1	2	4	39.5	-10	x
Larger Number	39	28	36	.5	50	y
Product	39	56	144	19.75	-500	xy

Primary

$$P = xy$$
$$P = x(40 - x)$$
$$P = 40x - x^2$$

$$40x - x^2 = 0$$
$$40 = 2x$$
$$x = 20, y = 20$$

Secondary

$$x + y = 40$$
$$y = 40 - x$$

Example 2) A rectangle has a perimeter of 71 feet. What length and width should it have so that its area is a maximum? What is this maximum area?

Width	10	5	1	.5	.1	x
Length	25.5	30.5	34.5	30	34.9	y
Area	255	152	34.5	15	3.49	xy

Primary

$$P = xy$$
$$P = x(35.5 - x)$$
$$P = 40x - x^2$$

$$35.5x - x^2 = 0$$
$$35.5 = 2x$$
$$x = 17.75 \text{ ft}, y = 17.75 \text{ ft}$$
$$\text{Area} = 315.063 \text{ ft}^2$$

Secondary

$$2x + 2y = 71$$
$$y = 35.5 - 2x$$

Example 3) Find two positive numbers that minimize the sum of twice the first number plus the second if the product of the two numbers is 288.

First Number	1	288	2	144	4	x
Second Number	288	1	144	2	72	y
Sum	290	577	148	290	80	$2x + y$

Primary

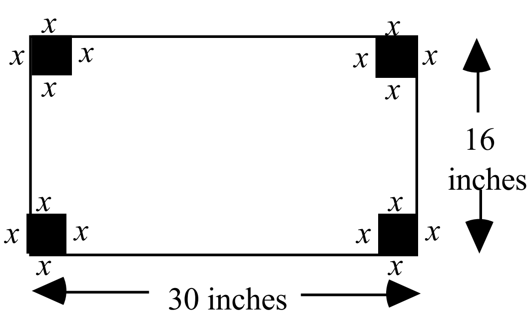
$$S = 2x + \tfrac{288}{x}$$
$$0 = 2 - \tfrac{288}{x^2}$$
$$2x^2 = 144 \Rightarrow x = 12$$

Secondary

$$xy = 288$$
$$y = \frac{288}{x}$$

Example 4) An open box is to be made from a piece of metal 16 by 30 inches by cutting out squares of equal size from the corners and bending up the sides. What size square should be cut out to create a box with greatest volume. What is the maximum volume as well?

Primary

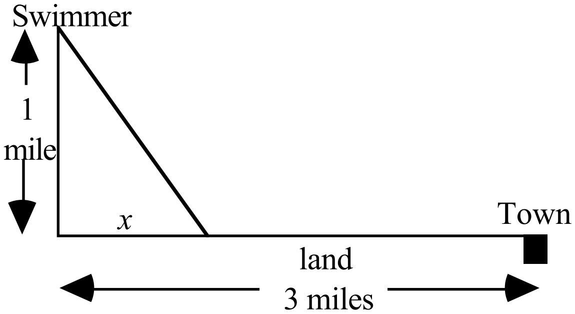


$$V = x(16 - 2x)(30 - 2x)$$
$$V = x(480 - 92x + 4x^2)$$
$$V = 480x - 92x^2 + 4x^3$$
$$0 = 480 - 184x + 12x^2$$

$$0 = 4(120 - 46x + 3x^2)$$
$$0 = 4(3x - 10)(x - 12)$$
$$x = \frac{10}{3} \text{ in}, x = 12(\text{impossible})$$
$$V = 725.926 \text{ in}^3$$

Example 5) I am 1 mile in the ocean and wish to get to a town 3 miles down the coast which is very rocky. I need to swim to the shore and then walk along the shore. What point should I swim to along the shoreline so that the time it takes to get to town is a minimum? I swim at 2 mph and walk at 4 mph.

Primary



$$T = \frac{\sqrt{x^2+1}}{2} + \frac{3-x}{4}$$

$$0 = \frac{x}{2\sqrt{x^2+1}} - \frac{1}{4}$$

$$2x = \sqrt{x^2+1}$$

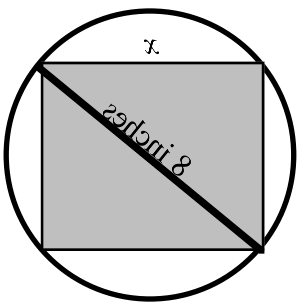
$$4x^2 = x^2 + 1$$

$$3x^2 = 1$$

$$x = \sqrt{\tfrac{1}{3}} = .577 \text{ miles}$$

Example 6) . Find the dimensions of the largest area rectangle which can be inscribed into a circle of radius 4 inches.

primary *secondary*



$$A = xy$$

$$A = x\left(64 - x^2\right)^{1/2}$$

$$0 = x\frac{-2x}{2\left(64-x^2\right)^{1/2}} + \left(64 - x^2\right)^{1/2}$$

$$\frac{x^2}{\left(64-x^2\right)^{1/2}} = \left(64-x^2\right)^{1/2}$$

$$x^2 = \left(64 - x^2\right)$$

$$2x^2 = 64 \Rightarrow x = 4\sqrt{2}, y = 4\sqrt{2} \text{ in}$$

$$x^2 + y^2 = 64$$

$$y = \left(64 - x^2\right)^{1/2}$$

How would this problem change if the radius were *r* inches?

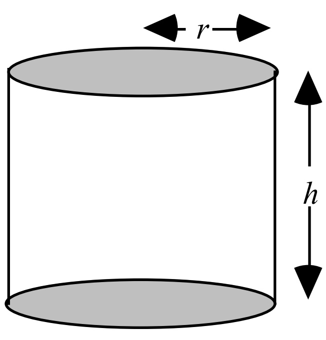
Change all the occurrences of 64 to $(2r)^2$ or $4r^2$- hence the final answer is a square with a side of $r\sqrt{2}$ inches.

Example 7) A 6-oz. can of Friskies Cat food contains a volume of approximately 14.5 cubic inches. How should the can be constructed so that the material made to make the can is a minimum?

Primary *Secondary*

Surface Area = Area of side + Area of Top & bottom $\quad V = \pi r^2 h$

$S = 2\pi r h + 2\pi r^2$



$$0 = 2\pi r\left(\frac{14.5}{\pi r^2}\right) + 2\pi r^2$$

$$0 = \frac{29}{r} + 4\pi r$$

$$\frac{29}{r^2} = 4\pi r$$

$$r^3 = \frac{29}{4\pi} = 1.321 \text{ in}, \; h = 2.643 \text{ in}$$

$$V = \pi r^2 h$$

$$14.5 = \pi r^2 h$$

$$h = \frac{14.5}{\pi r^2}$$