D. Even and Odd Functions

Functions that are even have the characteristic that for all a, $f(-a)=f(a)$. What this says is that plugging in a positive number a into the function or a negative number $-a$ into the function makes no difference … you will get the same result. Even functions are symmetric to the y-axis.

Functions that are odd have the characteristic that for all a, $f(-a)=-f(a)$. What this says is that plugging in a negative number $-a$ into the function will give you the same result as plugging in the positive number and taking the negative of that. So, odd functions are symmetric to the origin. If a graph is symmetric to the x-axis, it is not a function because it fails the vertical-line test.

1. Of the common functions in section 3, which are even, which are odd, and which are neither?

> Even: $y=a,\ y=x^2,\ y=|x|,\ y=\cos x$ Odd: $y=x,\ y=x^3,\ y=\dfrac{1}{x},\ y=\sin x$
>
> Neither: $y=\sqrt{x},\ y=\ln x,\ y=e^x,\ y=e^{-x}$

2. Show that the following functions are even:

a) $f(x)=x^4-x^2+1$

> $f(-x)=(-x)^4-(-x)^2+1$
> $=x^4-x^2+1=f(x)$

b) $f(x)=\left|\dfrac{1}{x}\right|$

> $f(-x)=\left|\dfrac{1}{-x}\right|=\left|\dfrac{1}{x}\right|=f(x)$

c) $f(x)=x^{2/3}$

> $f(-x)=(-x)^{2/3}=\left(\sqrt[3]{-x}\right)^2$
> $=\left(\sqrt[3]{x}\right)^2=f(x)$

3. Show that the following functions are odd:

a) $f(x)=x^3-x$

> $f(-x)=(-x)^3+x$
> $=x-x^3=-f(x)$

b) $f(x)=\sqrt[3]{x}$

> $f(-x)=\sqrt[3]{-x}=-\sqrt[3]{x}=-f(x)$

c) $f(x)=e^x-e^{-x}$

> $f(-x)=e^{-x}-e^x=-\left(e^x-e^{-x}\right)=-f(x)$

4. Determine if $f(x)=x^3-x^2+x-1$ is even, odd, or neither. Justify your answer.

> $f(-x)=-x^3-x^2-x-1\neq f(x)$ so f is not even. $-f(x)=-x^3+x^2-x+1\neq f(-x)$ so f is not odd.

Graphs may not be functions and yet have x-axis or y-axis or both. Equations for these graphs are usually expressed in "implicit form" where it is not expressed as "$y=$" or "$f(x)=$". If the equation does not change after making the following replacements, the graph has these symmetries:

x-axis: y with $-y$ y-axis: x with $-x$ origin: both x with $-x$ and y with $-y$

5. Determine the symmetry for $x^2+xy+y^2=0$.

> $x-$axis: $x^2+x(-y)+(-y)^2=0\Rightarrow x^2-xy+y^2=0$ so not symmetric to $x-$axis
>
> $y-$axis: $(-x)^2+(-x)(y)+y^2=0\Rightarrow x^2-xy+y^2=0$ so not symmetric to $y-$axis
>
> origin: $(-x)^2+(-x)(-y)+y^2=0\Rightarrow x^2+xy+y^2=0$ so symmetric to origin

D. Even and odd functions - Assignment

• Show work to determine if the following functions are even, odd, or neither.

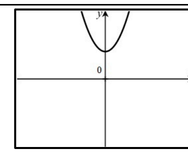
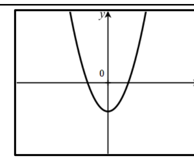
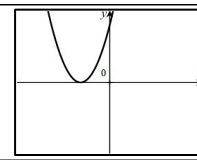
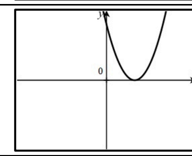
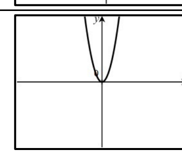
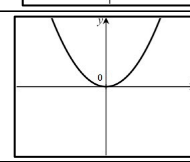
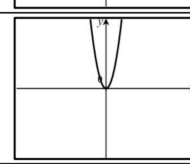
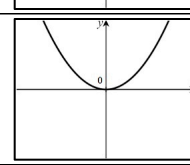
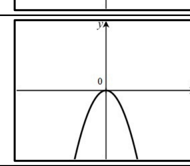
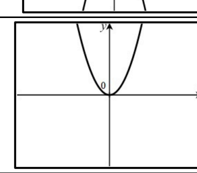
1. $f(x) = 7$
2. $f(x) = 2x^2 - 4x$
3. $f(x) = -3x^3 - 2x$
4. $f(x) = \sqrt{x+1}$
5. $f(x) = \sqrt{x^2+1}$
6. $f(x) = 8x$
7. $f(x) = 8x - \dfrac{1}{8x}$
8. $f(x) = |8x|$
9. $f(x) = |8x| - 8x$

Show work to determine if the graphs of these equations are symmetric to the *x*-axis*,* *y*-axis or the origin.

10. $4x = 1$
11. $y^2 = 2x^4 + 6$
12. $3x^2 = 4y^3$
13. $x = |y|$
14. $|x| = |y|$
15. $|x| = y^2 + 2y + 1$

E. Transformation of Graphs

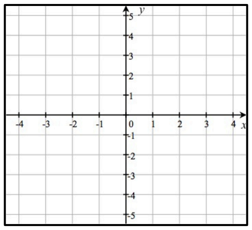
A curve in the form $y = f(x)$, which is one of the basic common functions from section C can be transformed in a variety of ways. The shape of the resulting curve stays the same but zeros and *y*-intercepts might change and the graph could be reversed. The table below describes transformations to a general function $y = f(x)$ with the parabolic function $f(x) = x^2$ as an example.

Notation	How $f(x)$ changes	Example with $f(x) = x^2$
$f(x)+a$	Moves graph up *a* units	
$f(x)-a$	Moves graph down *a* units	
$f(x+a)$	Moves graph *a* units left	
$f(x-a)$	Moves graph *a* units right	
$a \cdot f(x)$	$a > 1$: Vertical Stretch	
$a \cdot f(x)$	$0 < a < 1$: Vertical shrink	
$f(ax)$	$a > 1$: Horizontal compress (same effect as vertical stretch)	
$f(ax)$	$0 < a < 1$: Horizontal elongated (same effect as vertical shrink)	
$-f(x)$	Reflection across *x*-axis	
$f(-x)$	Reflection across *y*-axis	

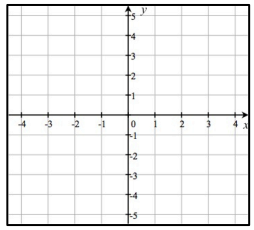
E. Transformation of Graphs Assignment

- Sketch the following equations:

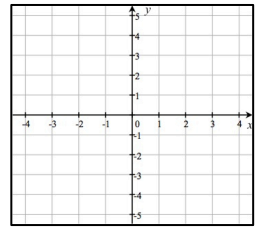
1. $y = -x^2$



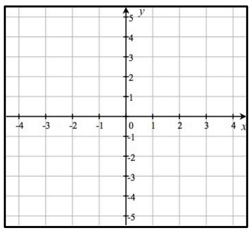
2. $y = 2x^2$



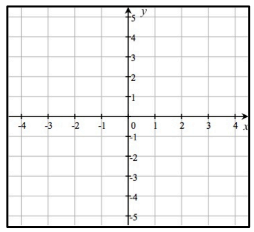
3. $y = (x-2)^2$



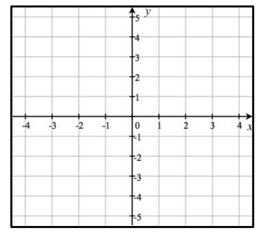
4. $y = 2 - \sqrt{x}$



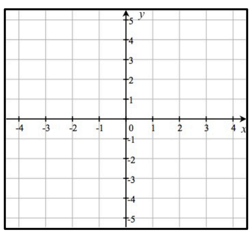
5. $y = \sqrt{x+1} + 1$



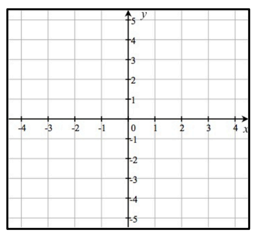
6. $y = \sqrt{4x}$



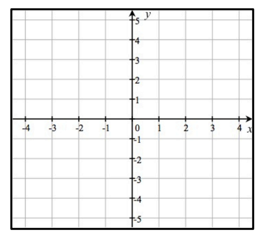
7. $y = |x+1| - 3$



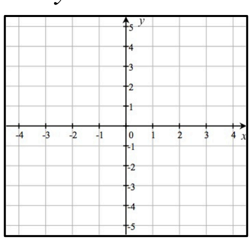
8. $y = -2|x-1| + 4$



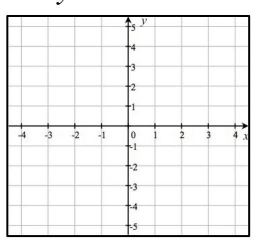
9. $y = -\left|\frac{x}{2}\right| - 1$



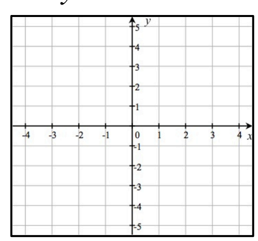
10. $y = 2^x - 2$



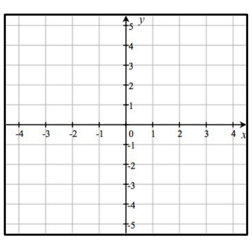
11. $y = -2^{x+2}$



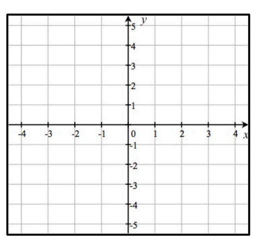
12. $y = 2^{-2x}$



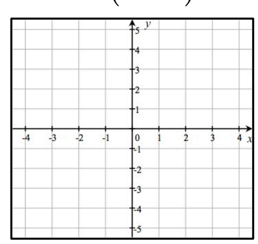
13. $y = \frac{1}{x-2}$



14. $y = \frac{-2}{x+1}$



15. $y = \frac{1}{(x+2)^2} - 3$



F. Special Factorization

While factoring skills were more important in the days when A topics were specifically tested, students still must know how to factor. The special forms that occur most regularly are:

Common factor: $x^3+x^2+x=x\left(x^2+x+1\right)$

Difference of squares: $x^2-y^2=(x+y)(x-y)$ or $x^{2n}-y^{2n}=\left(x^n+y^n\right)\left(x^n-y^n\right)$

Perfect squares: $x^2+2xy+y^2=(x+y)^2$

Perfect squares: $x^2-2xy+y^2=(x-y)^2$

Sum of cubes: $x^3+y^3=(x+y)\left(x^2-xy+y^2\right)$ - Trinomial unfactorable

Difference of cubes: $x^3-y^3=(x-y)\left(x^2+xy+y^2\right)$ - Trinomial unfactorable

Grouping: $xy+xb+ay+ab=x(y+b)+a(y+b)=(x+a)(y+b)$

The term "factoring" usually means that coefficients are rational numbers. For instance, x^2-2 could technically be factored as $\left(x+\sqrt{2}\right)\left(x-\sqrt{2}\right)$ but since $\sqrt{2}$ is not rational, we say that x^2-2 is not factorable. It is important to know that x^2+y^2 is unfactorable.

• Completely factor the following expressions.

1. $4a^2+2a$

   $\boxed{2a(a+2)}$

2. $x^2+16x+64$

   $\boxed{(x+8)^2}$

3. $4x^2-64$

   $\boxed{4(x+4)(x-4)}$

4. $5x^4-5y^4$

   $\boxed{5\left(x^2+1\right)(x+1)(x-1)}$

5. $16x^2-8x+1$

   $\boxed{(4x-1)^2}$

6. $9a^4-a^2b^2$

   $\boxed{a^2(3a+b)(3a-b)}$

7. $2x^2-40x+200$

   $\boxed{2(x-10)^2}$

8. x^3-8

   $\boxed{(x-2)\left(x^2+2x+4\right)}$

9. $8x^3+27y^3$

   $\boxed{(2x+3y)\left(4x^2-6xy+9y^2\right)}$

10. x^4+11x^2-80

    $\boxed{(x+4)(x-4)\left(x^2+5\right)}$

11. x^4-10x^2+9

    $\boxed{(x+1)(x-1)(x+3)(x-3)}$

12. $36x^2-64$

    $\boxed{4(3x+4)(3x-4)}$

13. x^3-x^2+3x-3

    $x^2(x-1)+3(x-1)$

    $(x-1)\left(x^2+3\right)$

14. $x^3+5x^2-4x-20$

    $x^2(x+5)-4(x+5)$

    $(x+5)(x-2)(x+2)$

15. $9-\left(x^2+2xy+y^2\right)$

    $9-(x+y)^2$

    $(3+x+y)(3-x-y)$

F. Special Factorization - Assignment

• Completely factor the following expressions

1. $x^3 - 25x$
2. $30x - 9x^2 - 25$
3. $3x^3 - 5x^2 + 2x$
4. $3x^8 - 3$
5. $16x^4 - 24x^2y + 9y^2$
6. $9a^4 - a^2b^2$
7. $4x^4 + 7x^2 - 36$
8. $250x^3 - 128$
9. $\dfrac{8x^3}{125} + \dfrac{64}{y^3}$
10. $x^5 + 17x^3 + 16x$
11. $144 + 32x^2 - x^4$
12. $16x^{4a} - y^{8a}$
13. $x^3 - xy^2 + x^2y - y^3$
14. $x^6 - 9x^4 - 81x^2 + 729$
15. $x^2 - 8xy + 16y^2 - 25$
16. $x^5 + x^3 + x^2 + 1$
17. $x^6 - 1$
18. $x^6 + 1$