G. Linear Functions

Probably the most important concept from precalculus that is required for differential calculus is that of linear functions. The formulas you need to know backwards and forwards are:

Slope: Given two points (x_1, y_1) and (x_2, y_2), the slope of the line passing through the points can be written as: $m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$.

Slope intercept form: the equation of a line with slope m and y-intercept b is given by $y = mx + b$.

Point-slope form: the equation of a line passing through the points (x_1, y_1) and slope m is given by $y - y_1 = m(x - x_1)$. While you might have preferred the simplicity of the $y = mx + b$ form in your algebra course, the $y - y_1 = m(x - x_1)$ form is far more useful in calculus.

Intercept form: the equation of a line with x-intercept a and y-intercept b is given by $\frac{x}{a} + \frac{y}{b} = 1$.

General form: $Ax + By + C = 0$ where A, B and C are integers. While your algebra teacher might have required your changing the equation $y - 1 = 2(x - 5)$ to general form $2x - y - 9 = 0$, you will find that on the AP calculus test, it is sufficient to leave equations for a lines in point-slope form and it is recommended not to waste time changing it unless you are specifically told to do so.

Parallel lines Two distinct lines are parallel if they have the same slope: $m_1 = m_2$.

Normal lines: Two lines are normal (perpendicular) if their slopes are negative reciprocals: $m_1 \cdot m_2 = -1$.

Horizontal lines have slope zero. **Vertical lines** have no slope (slope is undefined).

1. Find the equation of the line in slope-intercept form, with the given slope, passing through the given point.

a. $m = -4, (1,2)$

$\boxed{y - 2 = -4(x-1) \Rightarrow y = -4x + 6}$

b. $m = \frac{2}{3}, (-5,1)$

$\boxed{y - 1 = \frac{2}{3}(x-5) \Rightarrow y = \frac{2x}{3} - \frac{7}{3}}$

c. $m = 0, \left(-\frac{1}{2}, \frac{3}{4}\right)$

$\boxed{y = -\frac{3}{4}}$

2. Find the equation of the line in slope-intercept form, passing through the following points.

a. $(4,5)$ and $(-2,-1)$

$$\boxed{\begin{array}{l} m = \frac{5+1}{4+2} = 1 \\ y - 5 = x - 4 \Rightarrow y = x + 1 \end{array}}$$

b. $(0,-3)$ and $(-5,3)$

$$\boxed{\begin{array}{l} m = \frac{3+3}{-5-0} = \frac{-6}{5} \\ y + 3 = \frac{-6}{5}x \Rightarrow y = \frac{-6}{5}x - 3 \end{array}}$$

c. $\left(\frac{3}{4}, -1\right)$ and $\left(1, \frac{1}{2}\right)$

$$\boxed{\begin{array}{l} m = \left(\frac{\frac{1}{2}+1}{1-\frac{3}{4}}\right)\left(\frac{4}{4}\right) = \frac{2+4}{4-3} = 6 \\ y - \frac{1}{2} = 6(x-1) \Rightarrow y = 6x - \frac{11}{2} \end{array}}$$

3. Write equations of the line through the given point a) parallel and b) normal to the given line.

a. $(4,7)$, $4x - 2y = 1$

$$\boxed{\begin{array}{l} y = 2x - \frac{1}{2} \Rightarrow m = 2 \\ \text{a) } y - 7 = 2(x-4) \qquad \text{b) } y - 7 = \frac{-1}{2}(x-4) \end{array}}$$

b. $\left(\frac{2}{3}, 1\right)$, $x + 5y = 2$

$$\boxed{\begin{array}{l} y = \frac{-1}{5}x + 2 \Rightarrow m = \frac{-1}{5} \\ \text{a) } y - 1 = \frac{-1}{5}\left(x - \frac{2}{3}\right) \qquad \text{b) } y - 1 = 5\left(x - \frac{2}{3}\right) \end{array}}$$

G. Linear Functions - Assignment

1. Find the equation of the line in slope-intercept form, with the given slope, passing through the given point.

   a. $m = -7, (-3, -7)$ b. $m = \frac{-1}{2}, (2, -8)$ c. $m = \frac{2}{3}, \left(-6, \frac{1}{3}\right)$

2. Find the equation of the line in slope-intercept form, passing through the following points.

   a. $(-3, 6)$ and $(-1, 2)$ b. $(-7, 1)$ and $(3, -4)$ c. $\left(-2, \frac{2}{3}\right)$ and $\left(\frac{1}{2}, 1\right)$

3. Write equations of the line through the given point a) parallel and b) normal to the given line.

   a. $(5, -3),\ x + y = 4$ b. $(-6, 2),\ 5x + 2y = 7$ c. $(-3, -4),\ y = -2$

4. Find an equation of the line containing $(4, -2)$ and parallel to the line containing $(-1, 4)$ and $(2, 3)$. Put your answer in general form.

5. Find k if the lines $3x - 5y = 9$ and $2x + ky = 11$ are a) parallel and b) perpendicular.

H. Solving Quadratic Equations

Solving quadratics in the form of $ax^2+bx+c=0$ usually show up on the AP exam in the form of expressions that can easily be factored. But occasionally, you will be required to use the quadratic formula. When you have a quadratic equation, factor it, set each factor equal to zero and solve. If the quadratic equation doesn't factor or if factoring is too time-consuming, use the quadratic formula:

$x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$. The **discriminant** b^2-4ac will tell you how many solutions the quadratic has:

$$b^2-4ac\begin{cases}>0,\ 2\text{ real solutions (if a perfect square, the solutions are rational)}\\ =0,\ 1\text{ real solution}\\ <0,\ 0\text{ real solutions (or 2 imaginary solutions, but AP calculus does not deal with imaginaries)}\end{cases}$$

1. Solve for x.

a. $x^2+3x+2=0$

$$\boxed{\begin{array}{l}(x+2)(x+1)=0\\ x=-2, x=-1\end{array}}$$

b. $x^2-10x+25=0$

$$\boxed{\begin{array}{l}(x-5)^2=0\\ x=5\end{array}}$$

c. $x^2-64=0$

$$\boxed{\begin{array}{l}(x-8)(x+8)=0\\ x=8, x=-8\end{array}}$$

d. $2x^2+9x=18$

$$\boxed{\begin{array}{l}(2x-3)(x+6)=0\\ x=\dfrac{3}{2}, x=-6\end{array}}$$

e. $12x^2+23x=-10$

$$\boxed{\begin{array}{l}(4x+5)(3x+2)=0\\ x=-\dfrac{5}{4}, x=-\dfrac{2}{3}\end{array}}$$

f. $48x-64x^2=9$

$$\boxed{\begin{array}{l}(8x-3)^2=0\\ x=\dfrac{3}{8}\end{array}}$$

g. $x^2+5x=2$

$$\boxed{\begin{array}{l}x=\dfrac{-5\pm\sqrt{25+8}}{2}\\ x=\dfrac{-5\pm\sqrt{33}}{2}\end{array}}$$

h. $8x-3x^2=2$

$$\boxed{\begin{array}{l}x=\dfrac{8\pm\sqrt{64-24}}{6}\\ x=\dfrac{8\pm2\sqrt{10}}{6}=\dfrac{4\pm\sqrt{10}}{3}\end{array}}$$

i. $6x^2+5x+3=0$

$$\boxed{\begin{array}{l}x=\dfrac{-5\pm\sqrt{25-72}}{12}=\dfrac{-5\pm\sqrt{-47}}{12}\\ \text{No real solutions}\end{array}}$$

j. $x^3-3x^2+3x-9=0$

$$\boxed{\begin{array}{l}x^2(x-3)-3(x-3)=0\\ (x-3)(x^2-3)=0\\ x=3,\ x=\pm\sqrt{3}\end{array}}$$

k. $\dfrac{x}{3}-\dfrac{5}{2}=\dfrac{-3}{x}$

$$\boxed{\begin{array}{l}6x\left(\dfrac{x}{3}-\dfrac{5}{2}=\dfrac{-3}{x}\right)\\ 2x^2-15x+18=0\\ (2x-3)(x-6)=0\\ x=\dfrac{3}{2},\ x=6\end{array}}$$

l. $x^4-7x^2-8=0$

$$\boxed{\begin{array}{l}(x^2-8)(x^2+1)=0\\ x=\pm\sqrt{8}=\pm2\sqrt{2}\end{array}}$$

2. If $y=5x^2-3x+k$, for what values of k will the quadratic have two real solutions?

$$\boxed{(-3)^2-4(5)k>0\Rightarrow 9-20k>0\Rightarrow k<\dfrac{9}{20}}$$

H. Solving Quadratic Equations Assignment

1. Solve for x.

a. $x^2+7x-18=0$

b. $x^2+x+\frac{1}{4}=0$

c. $2x^2-72=0$

d. $12x^2-5x=2$

e. $20x^2-56x+15=0$

f. $81x^2+72x+16=0$

g. $x^2+10x=7$

h. $3x-4x^2=-5$

i. $7x^2-7x+2=0$

j. $x+\frac{1}{x}=\frac{17}{4}$

k. $x^3-5x^2+5x-25=0$

l. $2x^4-15x^3+18x^2=0$

2. If $y=x^2+kx-k$, for what values of k will the quadratic have two real solutions?

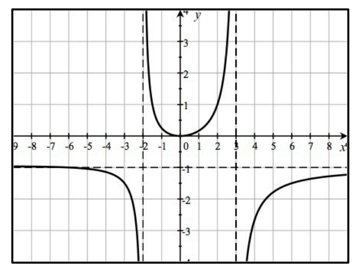
3. Find the domain of $y=\dfrac{2x-1}{6x^2-5x-6}$.

I. Asymptotes

Rational functions in the form of $y = \dfrac{p(x)}{q(x)}$ possibly have vertical asymptotes, lines that the graph of the curve approach but never cross. To find the **vertical asymptotes**, factor out any common factors of numerator and denominator, reduce if possible, and then set the denominator equal to zero and solve.

Horizontal asymptotes are lines that the graph of the function approaches when *x* gets very large or very small. While you learn how to find these in calculus, a rule of thumb is that if the highest power of *x* is in the denominator, the horizontal asymptote is the line $y = 0$. If the highest power of *x* is both in numerator and denominator, the horizontal asymptote will be the line $y = \dfrac{\text{highest degree coefficient in numerator}}{\text{highest degree coefficient in denominator}}$. If the highest power of *x* is in the numerator, there is no horizontal asymptote, but a slant asymptote which is not used in calculus.

1) Find any vertical and horizontal asymptotes for the graph of $y = \dfrac{-x^2}{x^2 - x - 6}$.

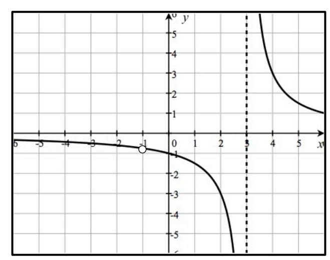


$$y = \frac{-x^2}{x^2 - x - 6} = \frac{-x^2}{(x-3)(x+2)}$$

Vertical asymptotes: $x - 3 = 0 \Rightarrow x = 3$ and $x + 2 = 0 \Rightarrow x = -2$
Horizontal asymptotes: Since the highest power of *x* is 2 in both numerator and denominator, there is a horizontal asymptote at $y = -1$.

This is confirmed by the graph to the right. Note that the curve actually crosses its horizontal asymptote on the left side of the graph.

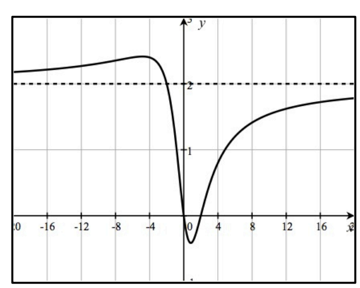
2) Find any vertical and horizontal asymptotes for the graph of $y = \dfrac{3x+3}{x^2 - 2x - 3}$.



$$y = \frac{3x+3}{x^2 - 2x - 3} = \frac{3(x+1)}{(x-3)(x+1)} = \frac{3}{x-3}$$

Vertical asymptotes: $x - 3 = 0 \Rightarrow x = 3$. Note that since the $(x+1)$ cancels, there is no vertical asymptote at $x = 1$, but a hole (sometimes called a removable discontinuity) in the graph.
Horizontal asymptotes: Since there the highest power of *x* is in the denominator, there is a horizontal asymptote at $y = 0$ (the *x*-axis). This is confirmed by the graph to the right.

3) Find any vertical and horizontal asymptotes for the graph of $y = \dfrac{2x^2 - 4x}{x^2 + 4}$.



$$y = \frac{2x^2 - 4x}{x^2 + 4} = \frac{2x(x-2)}{x^2 + 4}$$

Vertical asymptotes: None. The denominator doesn't factor and setting it equal to zero has no solutions.
Horizontal asymptotes: Since the highest power of *x* is 2 in both numerator and denominator, there is a horizontal asymptote at $y = 2$. This is confirmed by the graph to the right.

I. Asymptotes - Assignment

• Find any vertical and horizontal asymptotes and if present, the location of holes, for the graph of

1. $y = \dfrac{x-1}{x+5}$

2. $y = \dfrac{8}{x^2}$

3. $y = \dfrac{2x+16}{x+8}$

4. $y = \dfrac{2x^2+6x}{x^2+5x+6}$

5. $y = \dfrac{x}{x^2-25}$

6. $y = \dfrac{x^2-5}{2x^2-12}$

7. $y = \dfrac{4+3x-x^2}{3x^2}$

8. $y = \dfrac{5x+1}{x^2-x-1}$

9. $y = \dfrac{1-x-5x^2}{x^2+x+1}$

10. $y = \dfrac{x^3}{x^2+4}$

11. $y = \dfrac{x^3+4x}{x^3-2x^2+4x-8}$

12. $y = \dfrac{10x+20}{x^3-2x^2-4x+8}$

13. $y = \dfrac{1}{x} - \dfrac{x}{x+2}$ (hint: express with a common denominator)

J. Negative and Fractional Exponents

In calculus, you will be required to perform algebraic manipulations with **negative exponents** as well as **fractional exponents**. You should know the definition of a negative exponent: $x^{-n} = \dfrac{1}{x^n}, x \neq 0$. Note that negative powers do not make expressions negative; they create fractions. Typically expressions in multiple-choice answers are written with positive exponents and students are required to eliminate negative exponents. Fractional exponents create roots. The definition of $x^{1/2} = \sqrt{x}$ and $x^{a/b} = \sqrt[b]{x^a} = \left(\sqrt[b]{x}\right)^a$.

As a reminder: when we multiply, we add exponents: $\left(x^a\right)\left(x^b\right) = x^{a+b}$.

When we divide, we subtract exponents: $\dfrac{x^a}{x^b} = x^{a-b}, x \neq 0$

When we raise powers, we multiply exponents: $\left(x^a\right)^b = x^{ab}$

In your algebra course, leaving an answer with a radical in the denominator was probably not allowed. You had to rationalize the denominator: $\dfrac{1}{\sqrt{x}}$ changed to $\left(\dfrac{1}{\sqrt{x}}\right)\left(\dfrac{\sqrt{x}}{\sqrt{x}}\right) = \dfrac{\sqrt{x}}{x}$. In calculus, you will find that it is not necessary to rationalize and it is recommended that you not take the time to do so.

• Simplify and write with positive exponents. Note: # 12 involves complex fractions, covered in section K.

1. $-8x^{-2}$

$$\boxed{\frac{-8}{x^2}}$$

2. $\left(-5x^3\right)^{-2}$

$$\boxed{(-5)^{-2}x^{-6} = \frac{1}{(-5)^2 x^6} = \frac{1}{25x^6}}$$

3. $\left(\dfrac{-3}{x^4}\right)^{-2}$

$$\boxed{\frac{(-3)^{-2}}{\left(x^4\right)^{-2}} = \frac{1}{(-3)^2 x^{-8}} = \frac{x^8}{9}}$$

4. $\left(36x^{10}\right)^{1/2}$

$$\boxed{6x^5}$$

5. $\left(27x^3\right)^{-2/3}$

$$\boxed{\frac{1}{\left(27x^3\right)^{2/3}} = \frac{1}{9x^2}}$$

6. $\left(16x^{-2}\right)^{3/4}$

$$\boxed{16^{3/4}x^{-4/3} = \frac{8}{x^{4/3}}}$$

7. $\left(x^{1/2} - x\right)^{-2}$

$$\boxed{\frac{1}{\left(x^{1/2}-x\right)^2} = \frac{1}{x - 2x^{3/2} + x^2}}$$

8. $\left(4x^2 - 12x + 9\right)^{-1/2}$

$$\boxed{\frac{1}{\left[(2x-3)^2\right]^{1/2}} = \frac{1}{2x-3}}$$

9. $\left(x^{1/3}\right)\left(\dfrac{1}{2}x^{-1/2}\right) + \left(x^{1/2}+1\right)\left(\dfrac{1}{3}x^{-1/3}\right)$

$$\boxed{\frac{x^{1/3}}{2x^{1/2}} + \frac{x^{1/2}+1}{3x^{1/3}} = \frac{1}{2x^{1/6}} + \frac{x^{1/2}+1}{3x^{1/3}}}$$

10. $\dfrac{-2}{3}(8x)^{-5/3}(8)$

$$\boxed{\frac{-16}{3(8x)^{5/3}} = \frac{-16}{3(32)x^{5/3}} = -\frac{1}{6x^{5/3}}}$$

11. $\dfrac{(x+4)^{1/2}}{(x-4)^{-1/2}}$

$$\boxed{(x+4)^{1/2}(x-4)^{1/2} = \left(x^2-16\right)^{1/2}}$$

12. $\left(x^{-1} + y^{-1}\right)^{-1}$

$$\boxed{\left(\frac{1}{\frac{1}{x}+\frac{1}{y}}\right)\left(\frac{xy}{xy}\right) = \frac{xy}{y+x}}$$

J. Negative and Fractional Exponents - Assignment

Simplify and write with positive exponents.

1. $-12^2 x^{-5}$
2. $\left(-12x^5\right)^{-2}$
3. $\left(4x^{-1}\right)^{-1}$
4. $\left(\dfrac{-4}{x^4}\right)^{-3}$
5. $\left(\dfrac{5x^3}{y^2}\right)^{-3}$
6. $\left(x^3-1\right)^{-2}$
7. $\left(121x^8\right)^{1/2}$
8. $\left(8x^2\right)^{-4/3}$
9. $\left(-32x^{-5}\right)^{-3/5}$
10. $(x+y)^{-2}$
11. $\left(x^3+3x^2+3x+1\right)^{-2/3}$
12. $x\left(x^{1/2}-x\right)^{-2}$
13. $\dfrac{1}{4}\left(16x^2\right)^{-3/4}(32x)$
14. $\dfrac{\left(x^2-1\right)^{-1/2}}{\left(x^2+1\right)^{1/2}}$
15. $\left(x^{-2}+2^{-2}\right)^{-1}$

K. Eliminating Complex Fractions

Calculus frequently uses **complex fractions**, which are fractions within fractions. Answers are never left with complex fractions and they must be eliminated. There are two methods to eliminate complex fractions:

When the problem is in the form of $\frac{\frac{a}{b}}{\frac{c}{d}}$, we can "flip the denominator" and write it as $\frac{a}{b}\cdot\frac{d}{c}=\frac{ad}{bc}$.
However, this does not work when the numerator and denominator are not single fractions. The best way is to eliminate the complex fractions in all cases is to find the LCD (lowest common denominator) of all the fractions in the complex fraction. Multiply all terms by this LCD and you are left with a fraction that is magically no longer complex. **Important:** Note that $\frac{x^{-1}}{y^{-1}}$ can be written as $\frac{y}{x}$ but $\frac{1+x^{-1}}{y^{-1}}$ must be written as $\frac{1+\frac{1}{x}}{\frac{1}{y}}$.

• Eliminate the complex fractions.

1. $\dfrac{\frac{2}{3}}{\frac{5}{6}}$

$$\boxed{\left(\frac{\frac{2}{3}}{\frac{5}{6}}\right)\left(\frac{6}{6}\right)=\frac{4}{5}}$$

2. $\dfrac{1+\frac{2}{3}}{1+\frac{5}{6}}$

$$\boxed{\left(\frac{1+\frac{2}{3}}{1+\frac{5}{6}}\right)\left(\frac{6}{6}\right)=\frac{6+4}{6+5}=\frac{10}{11}}$$

3. $\dfrac{\frac{3}{4}+\frac{5}{3}}{2-\frac{1}{6}}$

$$\boxed{\left(\frac{\frac{3}{4}+\frac{5}{3}}{2-\frac{1}{6}}\right)\left(\frac{12}{12}\right)=\frac{9+20}{24-2}=\frac{29}{22}}$$

4. $\dfrac{1+\frac{1}{2}x^{-1}}{1+\frac{1}{3}x^{-1}}$

$$\boxed{\left(\frac{1+\frac{1}{2x}}{1+\frac{1}{3x}}\right)\left(\frac{6x}{6x}\right)=\frac{6x+3}{6x+2}}$$

5. $\dfrac{x-\frac{1}{2x}}{x^2+\frac{1}{4x^2}}$

$$\boxed{\left(\frac{x-\frac{1}{2x}}{x^2+\frac{1}{4x^2}}\right)\left(\frac{4x^2}{4x^2}\right)=\frac{4x^3-2x}{4x^4+1}}$$

6. $\dfrac{\frac{2}{5}x^{5/3}}{\frac{5}{3}}$

$$\boxed{\left(\frac{\frac{2}{5}x^{5/3}}{\frac{5}{3}}\right)\left(\frac{15}{15}\right)=\frac{6x^{5/3}}{25}}$$

7. $\dfrac{x^{-3}+x}{x^{-2}+1}$

$$\boxed{\left(\frac{\frac{1}{x^3}+x}{\frac{1}{x^2}+1}\right)\left(\frac{x^3}{x^3}\right)=\frac{1+x^4}{x+x^3}}$$

8. $\dfrac{\frac{1}{2}(2x+5)^{-2/3}}{\frac{-2}{3}}$

$$\boxed{\left(\frac{\frac{1}{2}}{\frac{-2}{3}(2x+5)^{2/3}}\right)\frac{6}{6}=\frac{-3}{4(2x+5)^{2/3}}}$$

9. $\dfrac{(x-1)^{1/2}-\frac{x(x-1)^{-1/2}}{2}}{x-1}$

$$\boxed{\begin{array}{l}\left(\dfrac{(x-1)^{1/2}-\dfrac{x}{2(x-1)^{1/2}}}{x-1}\right)\left[\dfrac{2(x-1)^{1/2}}{2(x-1)^{1/2}}\right]\\ \dfrac{2(x-1)-x}{2(x-1)^{3/2}}=\dfrac{x-2}{2(x-1)^{3/2}}\end{array}}$$

K. Eliminating Complex Fractions - Assignment

• Eliminate the complex fractions.

1. $\dfrac{\frac{5}{8}}{\frac{-2}{3}}$

2. $\dfrac{4-\frac{2}{9}}{3+\frac{4}{3}}$

3. $\dfrac{2+\frac{7}{2}+\frac{3}{5}}{5-\frac{3}{4}}$

4. $\dfrac{x-\frac{1}{x}}{x+\frac{1}{x}}$

5. $\dfrac{1+x^{-1}}{1-x^{-2}}$

6. $\dfrac{x^{-1}+y^{-1}}{x+y}$

7. $\dfrac{x^{-2}+x^{-1}+1}{x^{-2}-x}$

8. $\dfrac{\frac{1}{3}(3x-4)^{-3/4}}{\frac{-3}{4}}$

9. $\dfrac{2x(2x-1)^{1/2}-2x^{2}(2x-1)^{-1/2}}{(2x-1)}$