

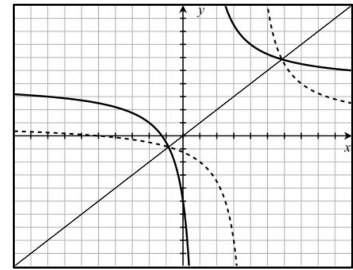
## L. Inverses

No topic in math confuses students more than inverses. If a function is a rule that maps  $x$  to  $y$ , an **inverse** is a rule that brings  $y$  back to the original  $x$ . If a point  $(x, y)$  is a point on a function  $f$ , then the point  $(y, x)$  is on the inverse function  $f^{-1}$ . Students mistakenly believe that since  $x^{-1} = \frac{1}{x}$ , then  $f^{-1} = \frac{1}{f}$ . This is decidedly incorrect.

If a function is given in equation form, to find the inverse, replace all occurrences of  $x$  with  $y$  and all occurrences of  $y$  with  $x$ . If possible, then solve for  $y$ . Using the “horizontal-line test” on the original function  $f$  will quickly determine whether or not  $f^{-1}$  is also a function. By definition,  $f(f^{-1}(x)) = x$ . The domain of  $f^{-1}$  is the range of  $f$  and the range of  $f^{-1}$  is the domain of  $f$ .

1. Find the inverse to  $y = \frac{4x+5}{x-1}$  and show graphically that its inverse is a function.

$$\text{Inverse: } x = \frac{4y+5}{y-1} \Rightarrow xy - x = 4y + 5 \Rightarrow y = \frac{x+5}{x-4}$$



Note that the function is drawn in bold and the inverse as dashed. The function and its inverse is symmetrical to the line  $y = x$ . The inverse is a function for two reasons: a) it passes the vertical line test or b) the function passes the horizontal line test.

2. Find the inverse to the following functions and show graphically that its inverse is a function.

a.  $y = 4x - 3$

$$\text{Inverse: } x = 4y - 3 \\ y = \frac{x+3}{4} \text{ (function)}$$

b.  $y = x^2 + 1$

$$\text{Inverse: } x = y^2 + 1 \\ y = \pm\sqrt{x-1} \text{ (not a function)}$$

c.  $y = x^2 + 4x + 4$

$$\text{Inverse: } x = y^2 + 4y + 4 \\ x = (y+2)^2 \Rightarrow \pm\sqrt{x} = y+2 \\ y = -2 \pm \sqrt{x} \text{ (not a function)}$$

3. Find the inverse to the following functions and show that  $f(f^{-1}(x)) = x$

a.  $f(x) = 7x + 4$

$$\text{Inverse: } x = 7y + 4 \\ y = f^{-1}(x) = \frac{x-4}{7} \\ f\left(\frac{x-4}{7}\right) = 7\left(\frac{x-4}{7}\right) + 4 = x$$

b.  $f(x) = \frac{1}{x-1}$

$$\text{Inverse: } x = \frac{1}{y-1} \Rightarrow xy - x = 1 \\ y = f^{-1}(x) = \frac{x+1}{x} \\ f\left(\frac{x+1}{x}\right) = \left(\frac{1}{\frac{x+1}{x}-1}\right)\left(\frac{x}{x}\right) \\ = \frac{x}{x+1-x} = x$$

c.  $f(x) = x^3 - 1$

$$\text{Inverse: } x = y^3 - 1 \\ y = f^{-1}(x) = \sqrt[3]{x+1} \\ f\left(\sqrt[3]{x+1}\right) = \left(\sqrt[3]{x+1}\right)^3 - 1 = x$$

4. Without finding the inverse, find the domain and range of the inverse to  $f(x) = \sqrt{x+2} + 3$ .

$$\text{Function: Domain: } [-2, \infty), \text{ Range: } [3, \infty) \quad \text{Inverse: Domain: } [3, \infty), \text{ Range: } [-2, \infty)$$

## L. Inverses – Assignment

1. Find the inverse to the following functions and show graphically that its inverse is a function.

a.  $2x - 6y = 1$

b.  $y = ax + b$

c.  $y = 9 - x^2$

d.  $y = \sqrt{1 - x^3}$

e.  $y = \frac{9}{x}$

f.  $y = \frac{2x + 1}{3 - 2x}$

2. Find the inverse to the following functions and show that  $f(f^{-1}(x)) = x$

a.  $f(x) = \frac{1}{2}x - \frac{4}{5}$

b.  $f(x) = x^2 - 4$

c.  $f(x) = \frac{x^2}{x^2 + 1}$

3. Without finding the inverse, find the domain and range of the inverse to  $f(x) = \frac{\sqrt{x+1}}{x^2}$ .

## M. Adding Fractions and Solving Fractional Equations

There are two major problem types with fractions: Adding/subtracting fractions and solving fractional equations. Algebra has taught you that in order to add fractions, you need to find an LCD and *multiply each fraction by one ...* in such a way that you obtain the LCD in each fraction. However, when you solve fractional equations (equations that involve fractions), you still find the LCD but you *multiply every term by the LCD*. When you do that, all the fractions disappear, leaving you with an equation that is hopefully solvable. Answers should be checked in the original equation.

1. a. Combine:  $\frac{x}{3} - \frac{x}{4}$

$$\text{LCD: } 12 \quad \frac{x}{3}\left(\frac{4}{4}\right) - \frac{x}{4}\left(\frac{3}{3}\right)$$

$$\frac{4x - 3x}{12} = \frac{x}{12}$$

b. Solve:  $\frac{x}{3} - \frac{x}{4} = 12$

$$12\left(\frac{x}{3}\right) - 12\left(\frac{x}{4}\right) = 12(12)$$

$$4x - 3x = 144 \Rightarrow x = 144$$

$$x = 144 : \frac{144}{3} - \frac{144}{4} = 48 - 36 = 12$$

2. a. Combine  $x + \frac{6}{x}$

$$\text{LCD: } x \quad x\left(\frac{x}{x}\right) + \frac{6}{x}$$

$$\frac{x^2 + 6}{x}$$

b. Solve:  $x + \frac{6}{x} = 5$

$$x(x) + x\left(\frac{6}{x}\right) = 5x$$

$$x^2 + 6 = 5x \Rightarrow x^2 - 5x + 6 = 0$$

$$(x-2)(x-3) = 0 \Rightarrow x = 2, x = 3$$

$$x = 2 : 2 + \frac{6}{2} = 2 + 3 = 5 \quad x = 3 : 3 + \frac{6}{3} = 3 + 2 = 5$$

3. a. Combine:  $\frac{12}{x+2} - \frac{4}{x}$

$$\text{LCD: } x(x+2) \quad \left(\frac{12}{x+2}\right)\left(\frac{x}{x}\right) - \frac{4}{x}\left(\frac{x+2}{x+2}\right)$$

$$\frac{12x - 4x - 8}{x(x+2)}$$

$$\frac{8x - 8}{x(x+2)}$$

b. Solve  $\frac{12}{x+2} - \frac{4}{x} = 1$

$$\frac{12}{x+2}(x)(x+2) - \frac{4}{x}(x)(x+2) = 1(x)(x+2)$$

$$12x - 4x - 8 = x^2 + 2x \Rightarrow x^2 - 6x + 8 = 0$$

$$(x-2)(x-4) = 0 \Rightarrow x = 2, 4$$

$$x = 2 : \frac{12}{4} - \frac{4}{2} = 3 - 2 = 1 \quad x = 4 : \frac{12}{6} - \frac{4}{4} = 2 - 1 = 1$$

4. a.  $\frac{x}{2x-6} - \frac{3}{x^2-6x+9}$

$$\text{LCD: } 2(x-3)^2$$

$$\frac{x}{2(x-3)}\left(\frac{x-3}{x-3}\right) - \frac{3}{(x-3)^2}\left(\frac{2}{2}\right)$$

$$\frac{x^2 - 3x - 6}{2(x-3)^2}$$

b. Solve  $\frac{x}{2x-6} - \frac{3}{x^2-6x+9} = \frac{x-2}{3x-9}$

$$\left[\frac{x}{2(x-3)} - \frac{3}{(x-3)^2} = \frac{x-2}{3(x-3)}\right] 6(x-3)^2$$

$$3x(x-3) - 18 = 2(x-3)(x-2)$$

$$3x^2 - 9x - 18 = 2x^2 - 10x + 12$$

$$x^2 + x - 30 = 0 \Rightarrow (x+6)(x-5) = 0 \Rightarrow x = -6, 5$$

$$x = -6 : \frac{-6}{-18} - \frac{3}{81} = \frac{-8}{-27} \quad x = 5 : \frac{5}{4} - \frac{3}{4} = \frac{3}{6}$$

## M. Adding Fractions and Solving Fractional Equations - Assignment

1. a. Combine:  $\frac{2}{3} - \frac{1}{x}$

b. Solve:  $\frac{2}{3} - \frac{1}{x} = \frac{5}{6}$

2. a. Combine:  $\frac{1}{x-3} + \frac{1}{x+3}$

b. Solve:  $\frac{1}{x-3} + \frac{1}{x+3} = \frac{10}{x^2-9}$

3. a. Combine:  $\frac{5}{2x} - \frac{5}{3x+15}$

b. Solve:  $\frac{5}{2x} - \frac{5}{3(x+5)} = \frac{5}{x}$

4. a. Combine:  $\frac{2x-1}{x-1} - \frac{3x}{2x+1}$

b. Solve:  $\frac{2x-1}{x-1} - \frac{3x}{2x+1} = \frac{x^2+11}{2x^2-x-1}$

## N. Solving Absolute Value Equations

Absolute value equations crop up in calculus, especially in BC calculus. The definition of the absolute value function is a piecewise function:  $f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$ . So, to solve an absolute value equation, split the absolute value equation into two equations, one with a positive parentheses and the other with a negative parentheses and solve each equation. It is possible that this procedure can lead to incorrect solutions so solutions must be checked.

• Solve the following equations.

1.  $|x - 1| = 3$

$x - 1 = 3$	$-(x - 1) = 3$
$x = 4$	$-x + 1 = 3$
	$x = -2$

2.  $|3x + 2| = 9$

$3x + 2 = 9$	$-(3x + 2) = 9$
$3x = 7$	$-3x - 2 = 9$
$x = \frac{7}{3}$	$3x = -11$
	$x = \frac{-11}{3}$

3.  $|2x - 1| - x = 5$

$2x - 1 - x = 5$	$-(2x - 1) - x = 5$
$x = 6$	$-3x = 4$
	$x = \frac{-4}{3}$

4.  $|x + 5| + 5 = 0$

$x + 5 + 5 = 0$	$-(x + 5) + 5 = 0$
$x = -10$	$-x - 5 + 5 = 0$
	$x = 0$

Both answers are invalid. It is impossible to add 5 to an absolute value and get 0.

5.  $|x^2 - x| = 2$

$(x^2 - x) = 2$	$-(x^2 - x) = 2$
$x^2 - x - 2 = 0$	$-x^2 + x = 2$
$(x - 2)(x + 1) = 0$	$0 = x^2 + x + 2$
$x = 2, x = -1$	No real solution

Both solutions check

6.  $|x - 10| = x^2 - 10x$

$x - 10 = x^2 - 10x$	$-(x - 10) = x^2 - 10x$
$x^2 - 11x + 10 = 0$	$-x + 10 = x^2 - 10x$
$(x - 1)(x - 10) = 0$	$x^2 - 9x - 10 = 0$
$x = 1, x = 10$	$(x - 10)(x + 1) = 0$
	$x = 10, x = -1$

Of the three solutions, only  $x = -1$  and  $x = 10$  are valid.

7.  $|x| + |2x - 2| = 8$

$x + 2x - 2 = 8$	$-x + 2x - 2 = 8$	$x - (2x - 2) = 8$	$-x - (2x - 2) = 8$
$3x = 10$	$x = 10$	$-x = 6$	$-3x = 6$
$x = \frac{10}{3}$		$x = -6$	$x = -2$

Of the four solutions, only  $x = \frac{10}{3}$  and  $x = -2$  are valid

## N. Solving Absolute Value Equations - Assignment

• Solve the following equations.

1.  $4|x+8|=20$

2.  $|1-7x|=13$

3.  $|8+2x|+2x=40$

4.  $|4x-5|+5x+2=0$

5.  $|x^2-2x-1|=7$

6.  $|12-x|=x^2-12x$

7.  $|x|+|4x-4|+x=14$

## O. Solving Inequalities

You may think that solving inequalities are just a matter of replacing the equal sign with an inequality sign. In reality, they can be more difficult and are fraught with dangers. And in calculus, inequalities show up more frequently than solving equations. Solving inequalities are a simple matter if they are based on linear equations. They are solved exactly like linear equations, remembering that if you multiply or divide both sides by a negative number, the direction of the inequality sign must be reversed.

However, if the inequality is more complex than a linear function, it is advised to bring all terms to one side. Pretend for a moment it is an equation and solve. Then create a number line which determines whether the transformed inequality is positive or negative in the intervals created on the number line and choose the correct intervals according to the inequality, paying attention to whether the zeroes are included or not.

If the inequality involves an absolute value, create two equations, replacing the absolute value with a positive parentheses and a negative parentheses and the inequality sign with an equal sign. Solve each, placing each solution on your number line. Then determine which intervals satisfy the original inequality.

If the inequality involves a rational function, set both numerator and denominator equal to zero, which will give you the values you need for your number line. Determine whether the inequality is positive or negative in the intervals created on the number line and choose the correct intervals according to the inequality, paying attention to whether the endpoints are included or not.

• Solve the following inequalities.

1.  $2x - 8 \leq 6x + 2$

$-10 \leq 4x$	or	$-4x \leq 10$
$\frac{-5}{2} \leq x$		$x \geq \frac{-5}{2}$

2.  $1 - \frac{3x}{2} > x - 5$

$2 - 3x > 2x - 10$
$12 > 5x \Rightarrow x < \frac{12}{5}$

3.  $-7 \leq 6x - 1 < 11$

$-6 \leq 6x \leq 12$
$-1 \leq x \leq 2$

4.  $|2x - 1| \leq x + 4$

$ 2x - 1  - x - 4 \leq 0$
$2x - 1 - x - 4 = 0 \quad -2x + 1 - x - 4 = 0$
$x = 5 \quad \quad \quad x = -1$
+++++0-----0+++++
-1                  5
So $-1 \leq x \leq 5$ or $[-1, 5]$

5.  $x^2 - 3x > 18$

$x^2 - 3x - 18 > 0 \Rightarrow (x + 3)(x - 6) > 0$
For $(x + 3)(x - 6) = 0$ , $x = -3, x = 6$
+++++0-----0+++++
-3                  6
So $x < -3$ or $x > 6$ or $(-\infty, -3) \cup (6, \infty)$

6.  $\frac{2x - 7}{x - 5} \leq 1$

$\frac{2x - 7}{x - 5} - 1 = 0 \Rightarrow \frac{2x - 7}{x - 5} - \frac{x - 5}{x - 5} < 0 \Rightarrow \frac{x - 2}{x - 5} < 0$
+++++0-----∞+++++
2                  5
So $2 \leq x < 5$ or $[2, 5)$

7. Find the domain of  $\sqrt{32 - 2x^2}$

$2(4 + x)(4 - x) \geq 0$
-----0+++++0-----
-4                  4
So $-4 \leq x \leq 4$ or $[-4, 4]$

## O. Solving Inequalities - Assignment

• Solve the following inequalities.

1.  $5(x-3) \leq 8(x+5)$

2.  $4 - \frac{5x}{3} > -\left(2x + \frac{1}{2}\right)$

3.  $\frac{3}{4} > x+1 > \frac{1}{2}$

4.  $x+7 \geq |5-3x|$

5.  $(x+2)^2 < 25$

6.  $x^3 < 4x^2$

7.  $\frac{5}{x-6} \geq \frac{1}{x+2}$

8. Find the domain of  $\sqrt{\frac{x^2-x-6}{x-4}}$



## P. Exponential Functions and Logarithms

Calculus spends a great deal of time on exponential functions in the form of  $b^x$ . Don't expect that when you start working with them in calculus, your teacher will review them. So learn them now! Students must know that the definition of a **logarithm** is based on exponential equations. If  $y = b^x$  then  $x = \log_b y$ . So when you are trying to find the value of  $\log_2 32$ , state that  $\log_2 32 = x$  and  $2^x = 32$  and therefore  $x = 5$ .

If the base of a log statement is not specified, it is defined to be 10. When we asked for  $\log 100$ , we are solving the equation:  $10^x = 100$  and  $x = 2$ . The function  $y = \log x$  has domain  $(0, \infty)$  and range  $(-\infty, \infty)$ . In calculus, we primarily use logs with base  $e$ , which are called natural logs (ln). So finding  $\ln 5$  is the same as solving the equation  $e^x = 5$ . Students should know that the value of  $e = 2.71828\dots$

There are three rules that students must keep in mind that will simplify problems involving logs and natural logs. These rules work with logs of any base including natural logs.

$$\text{i. } \log a + \log b = \log(a \cdot b) \qquad \text{ii. } \log a - \log b = \log\left(\frac{a}{b}\right) \qquad \text{iii. } \log a^b = b \log a$$

1. Find a.  $\log_4 8$

$$\begin{aligned} \log_4 8 &= x \\ 4^x &= 8 \Rightarrow 2^{2x} = 2^3 \\ x &= \frac{3}{2} \end{aligned}$$

b.  $\ln \sqrt{e}$

$$\begin{aligned} \ln \sqrt{e} &= x \\ e^x &= e^{1/2} \\ x &= \frac{1}{2} \end{aligned}$$

c.  $10^{\log 4}$

$$\begin{aligned} \log 4 &= x \\ 10^x &= 4 \text{ so } 10^{\log 4} = 4 \\ 10 \text{ to a power and log are inverses} \end{aligned}$$

d.  $\log 2 + \log 50$

$$\begin{aligned} \log(2 \cdot 50) &= \log 100 \\ 2 \end{aligned}$$

e.  $\log_4 192 - \log_4 3$

$$\begin{aligned} \log_4 \left(\frac{192}{3}\right) \\ \log_4 64 &= 3 \end{aligned}$$

f.  $\ln \sqrt[5]{e^3}$

$$\ln e^{3/5} = \frac{3}{5} \ln e = \frac{3}{5}$$

2. Solve a.  $\log_9(x^2 - x + 3) = \frac{1}{2}$

$$\begin{aligned} x^2 - x + 3 &= 9^{1/2} \\ x(x-1) &= 0 \\ x &= 0, x = 1 \end{aligned}$$

b.  $\log_{36} x + \log_{36}(x-1) = \frac{1}{2}$

$$\begin{aligned} \log_{36} x(x-1) &= \frac{1}{2} \\ x(x-1) &= 36^{1/2} = 6 \\ x^2 - x - 6 &= 0 \\ (x-3)(x+2) &= 0 \\ \text{Only } x &= 3 \text{ is in the domain} \end{aligned}$$

c.  $\ln x - \ln(x-1) = 1$

$$\begin{aligned} \ln\left(\frac{x}{x-1}\right) &= 1 \\ \frac{x}{x-1} &= e \Rightarrow x = ex - e \\ x &= \frac{e}{e-1} \end{aligned}$$

d.  $5^x = 20$

$$\begin{aligned} \log(5^x) &= \log 20 \\ x \log 5 &= \log 20 \\ x &= \frac{\log 20}{\log 5} \text{ or } x = \frac{\ln 20}{\ln 5} \end{aligned}$$

e.  $e^{-2x} = 5$

$$\begin{aligned} \ln e^{-2x} &= \ln 5 \\ -2x &= \ln 5 \Rightarrow x = \frac{-\ln 5}{2} \end{aligned}$$

f.  $2^x = 3^{x-1}$

$$\begin{aligned} \log(2^x) &= \log(3^{x-1}) \\ x \log 2 &= (x-1) \log 3 \\ x \log 2 &= x \log 3 - \log 3 \Rightarrow x = \frac{\log 3}{\log 3 - \log 2} \end{aligned}$$

## P. Exponential Functions and Logarithms - Assignment

1. Find a.  $\log_2 \frac{1}{4}$

b.  $\log_8 4$

c.  $\ln \frac{1}{\sqrt[3]{e^2}}$

d.  $5^{\log_5 40}$

e.  $e^{\ln 12}$

f.  $\log_{12} 2 + \log_{12} 9 + \log_{12} 8$

g.  $\log_2 \frac{2}{3} + \log_2 \frac{3}{32}$

h.  $\log_{\frac{1}{3}} \frac{4}{3} - \log_{\frac{1}{3}} 12$

i.  $\log_3 (\sqrt{3})^5$

2. Solve a.  $\log_5 (3x-8) = 2$

b.  $\log_9 (x^2 - x + 3) = \frac{1}{2}$

c.  $\log(x-3) + \log 5 = 2$

d.  $\log_2 (x-1) + \log_2 (x+3) = 5$

e.  $\log_5 (x+3) - \log_5 x = 2$

f.  $\ln x^3 - \ln x^2 = \frac{1}{2}$

g.  $3^{x-2} = 18$

h.  $e^{3x+1} = 10$

i.  $8^x = 5^{2x-1}$