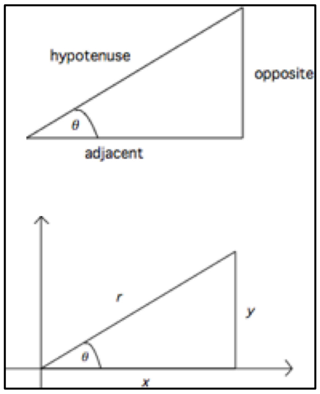
Q. Right Angle Trigonometry

Trigonometry is an integral part of AP calculus. Students must know the basic trig function definitions in terms of opposite, adjacent and hypotenuse as well as the definitions if the angle is in standard position.

Given a right triangle with one of the angles named θ, and the sides of the triangle relative to θ named opposite (y), adjacent (x), and hypotenuse (r) we define the 6 trig functions to be:



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r} \qquad \csc\theta = \frac{\text{hyptotenuse}}{\text{opposite}} = \frac{r}{y}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r} \qquad \sec\theta = \frac{\text{hyptotenuse}}{\text{adjacent}} = \frac{r}{x}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{x}{y}$$

The Pythagorean theorem ties these variables together: $x^2 + y^2 = r^2$. Students should recognize right triangles with integer sides: 3-4-5, 5-12-13, 8-15-17, 7-24-25. Also any multiples of these sides are also sides of a right triangle. Since r is the largest side of a right triangle, it can be shown that the range of $\sin\theta$ and $\cos\theta$ is $[-1,1]$, the range of $\csc\theta$ and $\sec\theta$ is $(-\infty,-1]\cup[1,\infty)$ and the range of $\tan\theta$ and $\cot\theta$ is $(-\infty,\infty)$.

Also vital to master is the signs of the trig functions in the four quadrants. A good way to remember this is A – S – T – C where All trig functions are positive in the 1st quadrant, Sin is positive in the 2nd quadrant, Tan is positive in the 3rd quadrant and Cos is positive in the 4th quadrant.

1. Let be a point on the terminal side of θ. Find the 6 trig functions of θ. (Answers need not be rationalized).

a) $P(-8,6)$

$x=-8, y=6, r=10$

$\sin\theta = \frac{3}{5}$ $\quad \csc\theta = \frac{5}{3}$

$\cos\theta = -\frac{4}{5}$ $\quad \sec\theta = -\frac{5}{4}$

$\tan\theta = -\frac{3}{4}$ $\quad \cot\theta = -\frac{4}{3}$

b. $P(1,3)$

$x=1,\ y=3,\ r=\sqrt{10}$

$\sin\theta = \frac{3}{\sqrt{10}}$ $\quad \csc\theta = \frac{\sqrt{10}}{3}$

$\cos\theta = \frac{1}{\sqrt{10}}$ $\quad \sec\theta = \sqrt{10}$

$\tan\theta = 3$ $\quad \cot\theta = \frac{1}{3}$

c. $P(-\sqrt{10},-\sqrt{6})$

$x=-\sqrt{10},\ y=-\sqrt{6},\ r=4$

$\sin\theta = -\frac{\sqrt{6}}{4}$ $\quad \csc\theta = -\frac{4}{\sqrt{6}}$

$\cos\theta = -\frac{\sqrt{10}}{4}$ $\quad \sec\theta = -\frac{4}{\sqrt{10}}$

$\tan\theta = \sqrt{\frac{3}{5}}$ $\quad \cot\theta = \sqrt{\frac{5}{3}}$

2. If $\cos\theta = \frac{2}{3}$, θ in quadrant IV, find $\sin\theta$ and $\tan\theta$

$x=2,\ r=3,\ y=-\sqrt{5}$

$\sin\theta = -\frac{\sqrt{5}}{3}, \tan\theta = -\frac{\sqrt{5}}{2}$

3. If $\sec\theta = \sqrt{3}$ find $\sin\theta$ and $\tan\theta$

θ is in quadrant I or IV

$x=1,\ y=\pm\sqrt{2},\ r=\sqrt{3}$

$\sin\theta = \pm\sqrt{\frac{2}{3}},\ \tan\theta = \pm\sqrt{2}$

4. Is $3\cos\theta + 4 = 2$ possible?

$3\cos\theta = -2$

$\cos\theta = -\frac{2}{3}$ which is possible.

Q. Right Angle Trigonometry - Assignment

1. Let be a point on the terminal side of θ. Find the 6 trig functions of θ. (Answers need not be rationalized).

a) $P(15,8)$

b. $P(-2,3)$

c. $P\left(-2\sqrt{5},-\sqrt{5}\right)$

2. If $\tan\theta = \dfrac{12}{5}$, θ in quadrant III, find $\sin\theta$ and $\cos\theta$

3. If $\csc\theta = \dfrac{6}{5}$, θ in quadrant II, find $\cos\theta$ and $\tan\theta$

4. $\cot\theta = \dfrac{-2\sqrt{10}}{3}$ find $\sin\theta$ and $\cos\theta$

5. Find the quadrants where the following is true: Explain your reasoning.

a. $\sin\theta > 0$ and $\cos\theta < 0$

b. $\csc\theta < 0$ and $\cot\theta > 0$

c. all functions are negative

6. Which of the following is possible? Explain your reasoning.

a. $5\sin\theta = -2$

b. $3\sin\alpha + 4\cos\beta = 8$

c. $8\tan\theta + 22 = 85$

R. Special Angles

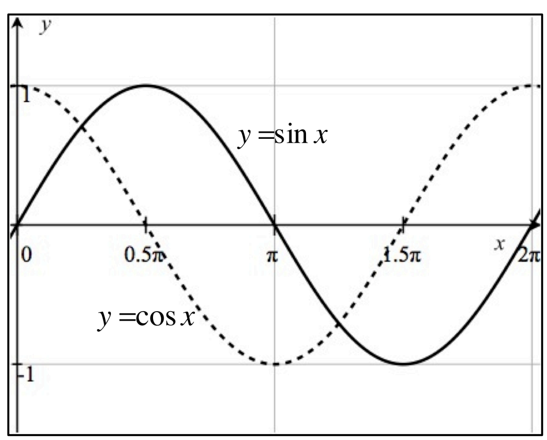
Students must be able to find trig functions of quadrant angles $(0,\ 90°, 180°, 270°)$ and special angles, those based on the $30° - 60° - 90°$ and $45° - 45° - 90°$ triangles.

First, for most calculus problems, angles are given and found in radians. Students must know how to convert degrees to radians and vice-versa. The relationship is 2π radians = 360° or π radians = 180°. Angles are assumed to be in radians so when an angle of $\frac{\pi}{3}$ is given, it is in radians. However, a student should be able to picture this angle as $\frac{180°}{3} = 60°$. It may be easier to think of angles in degrees than radians, but realize that unless specified, angle measurement must be written in radians. For instance, $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$.

The trig functions of **quadrant angles** $\left(0,\ 90°, 180°, 270° \text{ or } 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right)$ can quickly be found. Choose a point along the angle and realize that *r* is the distance from the origin to that point and always positive. Then use the definitions of the trig functions.

θ	point	x	y	r	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
0	$(1,0)$	1	0	1	0	1	0	does not exist	1	does not exist
$\frac{\pi}{2}$ or 90°	$(0,1)$	0	1	1	1	0	does not exist	1	does not exist	0
π or 180°	$(-1,0)$	-1	0	1	0	-1	0	does not exist	-1	does not exist
$\frac{3\pi}{2}$ or 270°	$(0,-1)$	0	-1	1	-1	0	Does not exist	-1	does not exist	0

If you picture the graphs of $y = \sin x$ and $y = \cos x$ as shown to the right, you need not memorize the table. You must know these graphs backwards and forwards.



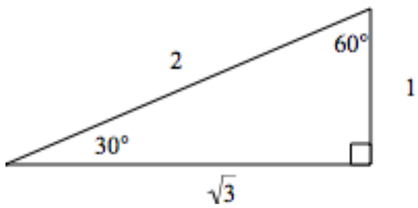
• Without looking at the table, find the value of

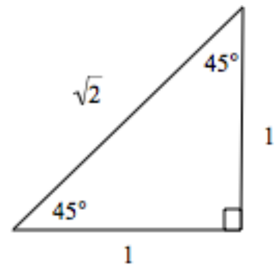
a. $5\cos 180° - 4\sin 270°$

$$\boxed{\begin{array}{l} 5(-1) - 4(-1) \\ -5 + 4 = -1 \end{array}}$$

b. $\left(\dfrac{8\sin\frac{\pi}{2} - 6\tan\pi}{5\sec\pi - \csc\frac{3\pi}{2}}\right)^2$ $\quad\boxed{\left[\dfrac{8(1) - 6(0)}{5(-1) - (-1)}\right]^2 = \left(\dfrac{8}{-4}\right)^2 = 4}$

Because over half of the AP exam does not use a calculator, you must be able to determine trig functions of **special angles**. You must know the relationship of sides in both $30°-60°-90°$ $\left(\frac{\pi}{6},\frac{\pi}{3},\frac{\pi}{2}\right)$ and $45°-45°-90°$ $\left(\frac{\pi}{4},\frac{\pi}{4},\frac{\pi}{2}\right)$ triangles.





In a $30°-60°-90°$ $\left(\frac{\pi}{6},\frac{\pi}{3},\frac{\pi}{2}\right)$ triangle, the ratio of sides is $1-\sqrt{3}-2$.

In a $45°-45°-90°$ $\left(\frac{\pi}{4},\frac{\pi}{4},\frac{\pi}{2}\right)$ triangle, the ratio of sides is $1-1-\sqrt{2}$.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
$30°\left(\text{or } \frac{\pi}{6}\right)$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$45°\left(\text{or } \frac{\pi}{4}\right)$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$60°\left(\text{or } \frac{\pi}{3}\right)$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

Special angles are any multiple of $30°\left(\frac{\pi}{6}\right)$ or $45°\left(\frac{\pi}{2}\right)$. To find trig functions of any of these angles, draw them and find the **reference angle** (the angle created with the *x*-axis). Although most problems in calculus will use radians, you might think easier using degrees. This will create one of the triangles above and trig functions can be found, remembering to include the sign based on the quadrant of the angle. Finally, if an angle is outside the range of $0°$ to $360°(0 \text{ to } 2\pi)$, you can always add or subtract $360°(2\pi)$ to find trig functions of that angle. These angles are called **co-terminal angles**. It should be pointed out that $390° \neq 30°$ but $\sin 390° = \sin 30°$.

• Find the exact value of the following

a. $4\sin 120° - 8\cos 570°$

Subtract $360°$ from $570°$
$4\sin 120° - 8\cos 210°$
$120°$ is in quadrant II with reference angle $60°$.
$210°$ is in quadrant III with reference angle $30°$.
$4\left(\frac{\sqrt{3}}{2}\right)-8\left(\frac{-\sqrt{3}}{2}\right)=6\sqrt{3}$

b. $\left(2\cos\pi - 5\tan\frac{7\pi}{4}\right)^2$

$(2\cos 180° - 5\tan 315°)^2$
$180°$ is a quadrant angle
$315°$ is in quadrant III with reference angle $45°$
$\left[2(-1)-5(-1)\right]^2 = 9$

R. Special Angles – Assignment

• Evaluate each of the following without looking at a chart.

1. $\sin^2 120° + \cos^2 120°$

2. $2\tan^2 300° + 3\sin^2 150° - \cos^2 180°$

3. $\cot^2 135° - \sin 210° + 5\cos^2 225$

4. $\cot(-30°) + \tan(600°) - \csc(-450°)$

5. $\left(\cos\dfrac{2\pi}{3} - \tan\dfrac{3\pi}{4}\right)^2$

6. $\left(\sin\dfrac{11\pi}{6} - \tan\dfrac{5\pi}{6}\right)\left(\sin\dfrac{11\pi}{6} + \tan\dfrac{5\pi}{6}\right)$

• Determine whether each of the following statements are true or false.

7. $\sin\dfrac{\pi}{6} + \sin\dfrac{\pi}{3} = \sin\left(\dfrac{\pi}{6} + \dfrac{\pi}{3}\right)$

8. $\dfrac{\cos\dfrac{5\pi}{3} + 1}{\tan^2\dfrac{5\pi}{3}} = \dfrac{\cos\dfrac{5\pi}{3}}{\sec\dfrac{5\pi}{3} - 1}$

9. $2\left(\dfrac{3\pi}{2} + \sin\dfrac{3\pi}{2}\right)\left(1 + \cos\dfrac{3\pi}{2}\right) > 0$

10. $\dfrac{\cos^3\dfrac{4\pi}{3} + \sin\dfrac{4\pi}{3}}{\cos^2\dfrac{4\pi}{3}} > 0$

S. Trigonometric Identities

Trig identities are equalities involving trig functions that are true for all values of the occurring angles. While you are not asked these identities specifically in calculus, knowing them can make some problems easier. The following chart gives the major trig identities that you should know. To prove trig identities, you usually start with the more involved expression and use algebraic rules and the fundamental trig identities. A good technique is to change all trig functions to sines and cosines.

Fundamental Trig Identities

$$\csc x=\frac{1}{\sin x},\quad \sec x=\frac{1}{\cos x},\quad \cot x=\frac{1}{\tan x},\quad \tan x=\frac{\sin x}{\cos x},\quad \cot x=\frac{\cos x}{\sin x}$$

$$\sin^2 x+\cos^2 x=1,\quad 1+\tan^2 x=\sec^2 x,\quad 1+\cot^2 x=\csc^2 x$$

Sum Identities

$$\sin(A+B)=\sin A\cos B+\cos A\sin B \qquad \cos(A+B)=\cos A\cos B-\sin A\sin B$$

Double Angle Identities

$$\sin(2x)=2\sin x\cos x \qquad \cos(2x)=\cos^2 x-\sin^2 x=1-2\sin^2 x=2\cos^2 x-1$$

• Verify the following identities.

1. $\left(\tan^2 x+1\right)\left(\cos^2-1\right)=-\tan^2 x$

$$\left(\sec^2 x\right)\left(-\sin^2 x\right)$$
$$\left(\frac{1}{\cos^2 x}\right)\left(-\sin^2 x\right)$$
$$-\tan^2 x$$

2. $\sec x-\cos x=\sin x\tan x$

$$\frac{1}{\cos x}-\cos x\left(\frac{\cos x}{\cos x}\right)$$
$$\frac{1-\cos^2 x}{\cos x}=\frac{\sin^2 x}{\cos x}$$
$$\sin x\left(\frac{\sin x}{\cos x}\right)=\sin x\tan x$$

3. $\dfrac{\cot^2 x}{1+\csc x}=\dfrac{1-\sin x}{\sin x}$

$$\left(\frac{\frac{\cos^2}{\sin^2 x}}{1+\frac{1}{\sin x}}\right)\frac{\sin^2 x}{\sin^2 x}=\frac{\cos^2 x}{\sin^2 x+\sin x}$$
$$\frac{1-\sin^2 x}{\sin x(1+\sin x)}=\frac{(1+\sin x)(1-\sin x)}{\sin x(1+\sin x)}$$
$$\frac{1-\sin x}{\sin x}$$

4. $\dfrac{1+\sin x}{\cos x}+\dfrac{\cos x}{1+\sin x}=2\sec x$

$$\left(\frac{1+\sin x}{\cos x}\right)\left(\frac{1+\sin x}{1+\sin x}\right)+\left(\frac{\cos x}{1+\sin x}\right)\left(\frac{\cos x}{\cos x}\right)$$
$$\frac{1+2\sin x+\sin^2+\cos^2 x}{\cos x(1+\sin x)}$$
$$\frac{1+2\sin x+1}{\cos x(1+\sin x)}=\frac{2+2\sin x}{\cos x(1+\sin x)}$$
$$\frac{2(1+\sin x)}{\cos x(1+\sin x)}=2\sec x$$

5. $\cos^4 2x-\sin^4 2x=\cos 4x$

$$\left(\cos^2 2x+\sin^2 2x\right)\left(\cos^2 2x-\sin^2 2x\right)$$
$$1\left[\cos 2(2x)\right]$$
$$\cos 4x$$

6. $\sin(3\pi-x)=\sin x$

$$\sin 3\pi\cos x-\cos 3\pi\sin x$$
$$0(\cos x)-(-1)\sin x=\sin x$$

S. Trig Identities – Assignment

• Verify the following identities.

1. $(1+\sin x)(1-\sin x)=\cos^2 x$

2. $\sec^2 x+3=\tan^2 x+4$

3. $\dfrac{1-\sec x}{1-\cos x}=-\sec x$

4. $\dfrac{1}{1+\tan x}+\dfrac{1}{1+\cot x}=1$

5. $\dfrac{\cos x-\cos y}{\sin x+\sin y}+\dfrac{\sin x-\sin y}{\cos x+\cos y}=0$

6. $\dfrac{\sin^3 x+\cos^3 x}{\sin x+\cos x}=1-\sin x\cos x$

7. $\csc 2x=\dfrac{\csc x}{2\cos x}$

8. $\dfrac{\cos 3x}{\cos x}=1-4\sin^2 x$

T. Solving Trig Equations and Inequalities

Trig equations are equations using trig functions. Typically they have many (or infinite) number of solutions so usually they are solved within a specific domain. Without calculators, answers are either quadrant angles or special angles, and again, they must be expressed in radians.

For trig inequalities, set both numerator and denominator equal to zero and solve. Make a sign chart with all these values included and examine the sign of the expression in the intervals. Basic knowledge of the sine and cosine curve is invaluable from section R is invaluable.

• Solve for x on $[0,2\pi)$

1. $x\cos x = 3\cos x$

Do not divide by $\cos x$ as you will lose solutions

$\cos x(x-3) = 0$

$\cos x = 0 \qquad x-3=0$

$x = \frac{\pi}{2}, \frac{3\pi}{2} \qquad x = 3$

You must work in radians.

Saying $x = 90°$ makes no sense.

2. $\tan x + \sin^2 x = 2 - \cos^2 x$

$\tan x + \sin^2 x + \cos^2 x = 2$

$\tan x + 1 = 2$

$\tan x = 1$

$x = \frac{\pi}{4}, \frac{5\pi}{4}$

Two answers as tangent is positive in quadrants I and III.

3. $3\tan^2 x - 1 = 0$

$3\tan^2 x = 1$

$\tan^2 x = \frac{1}{3}$

$\tan x = \pm\sqrt{\frac{1}{3}} = \pm\frac{\sqrt{3}}{3}$

$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

4. $3\cos x = 2\sin^2 x$

$3\cos x = 2(1-\cos^2 x)$

$2\cos^2 x + 3\cos x - 2 = 0$

$(2\cos x - 1)(\cos x + 2) = 0$

$2\cos x = 1 \qquad \cos x = -2$

$\cos x = \frac{1}{2} \qquad$ No solution

$x = \frac{\pi}{3}, \frac{5\pi}{3}$

7. Solve for x on $[0,2\pi)$: $\dfrac{2\cos x + 1}{\sin^2 x} > 0$

$2\cos x = -1 \Rightarrow \cos x = \frac{-1}{2} \Rightarrow x = \frac{2\pi}{3}, \frac{4\pi}{3}$

$\sin^2 x = 0 \Rightarrow x = 0, \pi$

0		$\frac{2\pi}{3}$		π		$\frac{4\pi}{3}$		2π
	$+++++++$	0	$-------$	∞	$-------$	0	$+++++++$	

Answer: $\left[0, \frac{2\pi}{3}\right) \cup \left(\frac{4\pi}{3}, 2\pi\right)$

T. Solving Trig Equations and Inequalities - Assignment

• Solve for x on $[0, 2\pi)$

1. $\sin^2 x = \sin x$

2. $3\tan^3 x = \tan x$

3. $\sin^2 x = 3\cos^2 x$

4. $\cos x + \sin x \tan x = 2$

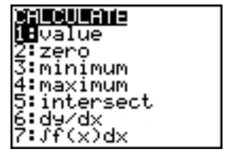
5. $\sin x = \cos x$

6. $2\cos^2 x + \sin x - 1 = 0$

7. Solve for x on $[0, 2\pi)$: $\dfrac{x - \pi}{\cos^2 x} < 0$

U. Graphical Solutions to Equations and Inequalities

You have a shiny new calculator. So when are we going to use it? So far, no mention has been made of it. Yet, the calculator is a tool that is required in the AP calculus exam. For about 25% of the exam, a calculator is permitted. So it is vital you know how to use it.



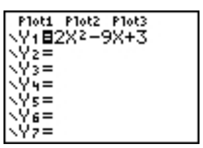
There are several settings on the calculator you should make. First, so you don't get into rounding difficulties, it is suggested that you set your calculator to three decimal places. That is a standard in AP calculus so it is best to get into the habit. To do so, press $\boxed{\text{MODE}}$ and on the 2nd line, take it off FLOAT and change it to 3. And second, set your calculator to radian mode from the MODE screen. There may be times you might want to work in degrees but it is best to work in radians.

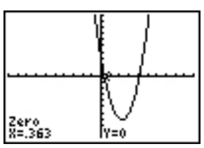
You must know how to graph functions. The best way to graph a function is to input the function using the $\boxed{\text{Y=}}$ key. Set your XMIN and XMAX using the $\boxed{\text{WINDOW}}$ key. Once you do that, you can see the graph's total behavior by pressing $\boxed{\text{ZOOM}}$ 0. To evaluate a function at a specific value of x, the easiest way to do so is to press these keys: $\boxed{\text{VARS}}$ $\boxed{\rightarrow}$ $\boxed{\text{1:Function 1}}$ $\boxed{\text{1:Y1}}$ $\boxed{(}$ and input your x-value.
Other than basic calculations, and taking trig functions of angles, there are three calculator features you need to know: evaluating functions at values of x and finding zeros of functions, which we know is finding where the function crosses the x-axis. The other is finding the point of intersection of two graphs. Both of these features are found on the TI-84+ calculator in the CALC menu $\boxed{\text{2}^{\text{ND}}}$ $\boxed{\text{TRACE}}$. They are 1:value, 2: zero, and 5: intersect.

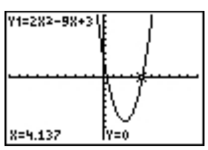
Solving equations using the calculator is accomplished by setting the equation equal to zero, graphing the function, and using the ZERO feature. To use it, press $\boxed{\text{2}^{\text{ND}}}$ $\boxed{\text{TRACE}}$ $\boxed{\text{ZERO}}$. You will be prompted to type in a number to the left of the approximate zero, to the right of the approximate zero, and a guess (for which you can press $\boxed{\text{ENTER}}$). You will then see the zero (the solution) on the screen.

• Solve these equations graphically.

1. $2x^2 - 9x + 3 = 0$

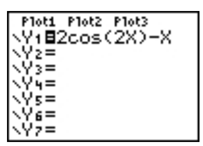


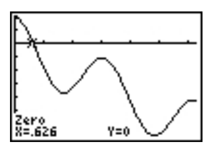


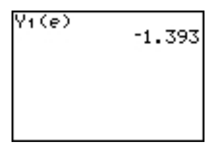


This equation can be solved with the quadratic formula.
$$x = \frac{9 \pm \sqrt{81-24}}{4} = \frac{9 \pm \sqrt{57}}{4}$$

2. $2\cos 2x - x = 0$ on $[0, 2\pi)$ and find $2\cos(2e) - e$.



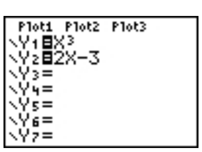


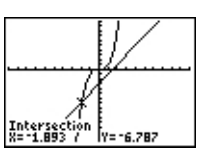


If this were the inequality $2\cos 2x - x > 0$, the answer would be $[0, 0.626)$.

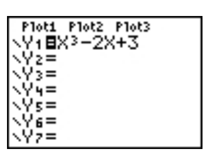
3. Find the x-coordinate of the intersection of $y = x^3$ and $y = 2x - 3$

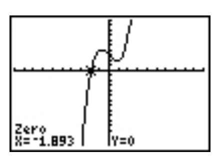
You can use the intersection feature.





Or set them equal to each other: $x^3 = 2x - 3$ or $x^3 - 2x + 3 = 0$





U. Graphical Solutions to Equations and Inequalities – Assignment

• Solve these equations or inequalities graphically.

1. $3x^3 - x - 5 = 0$

2. $x^3 - 5x^2 + 4x - 1 = 0$

3. $2x^2 - 1 = 2^x$

4. $2\ln(x+1) = 5\cos x$ on $[0, 2\pi)$

5. $x^4 - 9x^2 - 3x - 15 < 0$

6. $\dfrac{x^2 - 4x - 4}{x^2 + 1} > 0$ on $[0, 8]$

7. $x \sin x^2 > 0$ on $[0, 3]$

8. $\cos^{-1} x > x^2$ on $[-1, 1]$